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GEOMETRICAL OPTICS.

Cambridge:
PRINTED BY C. J. CLAY, M.A.
AT THE UNIVERSITY PRESS.

GEOMETRICAL OPTICS

ADAPTED TO THE USE OF THE HIGHER
CLASSES IN SCHOOLS, &c.

BY

OSMUND AIRY, B.A.

TRINITY COLLEGE, CAMBRIDGE.
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PREFACE.

THIS is, I imagine, the first time that any attempt has been made to adapt the subject of geometrical optics to the reading of the higher classes in our good schools. That this should be so is the more a matter for remark, since the subject would appear to be peculiarly fitted for such an adaptation. The great simplicity of its primary laws, in the first place, and the very small amount of analysis which, generally speaking, they involve, render the commencement not so unattractive as that of many subjects which are usually taken as the beginning of the second course of mathematical reading: whilst the new ideas and trains of thought which are introduced at every stage appear to me to be at once interesting and valuable. The conception of a virtual image, to take an early instance, is probably an entirely new one to the reader's mind. So also is the idea of a caustic curve, and the subject abounds with similar new considerations.

But the chief advantage that this subject possesses appears to me to be the middle position which it holds between the purely theoretical and the purely experimental. It contains sufficient of physical interest to give reality, and the easy and certain experiments by which it can be illustrated are convincing evidences of the correctness of the results, whilst the analysis which is requisite to obtain these results is sufficient to afford the reader, who up till now has been studying Algebra and Euclid, a proof that the said Algebra and Euclid have really some distinct use in explaining the phænomena of common life.

When the idea of writing such a book was first suggested to me by my old master, Professor Drew of King's College, I was afraid that there was hardly room, under the present system, for the subject in the course of a boy's school-work: I therefore, after making some progress in it, wrote to obtain the opinion of the head mathematical masters of two or three of our best schools; and I cannot sufficiently acknowledge the courtesy with which these gentlemen gave me the information I required. While acknowledging that the doubt which I had expressed existed in their own minds, they urged me strongly to continue my work, as they fully agreed in recognizing the great use of introducing a subject into school-work which should combine new ideas with practice in former knowledge.

The familiar use of the subject of Geometrical Optics by all who have received any but a very slight mathematical training precludes the possibility of originality in the facts themselves, and admits of but small originality in their treatment. At the same time it is hoped that it will be found that this book is not merely a rescript of any existing work.

It might be thought that it would be an easy task to adapt a subject of simple principles to the comprehension of boys possessing mathematical knowledge of the standard usually found in the higher classes of our schools. I imagined so myself, and during the course of my working I have been gradually learning my error. I can only wish that what I consider to be an object of the highest importance had been approached by some one more fitted both by mathematical knowledge and experience than by one whose sole qualifications are considerable fondness of the subject and a great desire to afford to beginners some glimpse of one of the steps to science to which their daily mathematics are leading. I have endeavoured as much as possible to avoid the example of those popular lecturers who explain difficulties by ignoring them. But as the nature of my design necessitated brevity, I have omitted entirely one or two portions of the subject which I considered unnecessary to a clear understanding of the rest, and which appear to me better learnt at a more advanced stage,—such are the subjects of *Illumination of Surfaces* a *Achromatism.*

I need not say that I shall feel most thank for notices of corrections that are needed, and any suggestions as to the improvement of the bo

O. AIRY.

WELLINGTON COLLEGE,
April 2, 1870.

ERRATA.

Page 17, fig. 12, lowest line from Q is unnecessary.

37, fig. 24 (ii), $\left.\begin{array}{l} q_1' \text{ should be } q_1 \\ q_1 \cdots\cdots\cdots\cdots q_1' \end{array}\right\}$, and similarly on page 39.

40, fig. 25 (3 and 4), cross line of arrow slightly too long.

46, line 7, *for* O *read* Q.

— line 5 from foot, *for* fig. 29 *read* fig. 28.

54, line 7 from foot, insert *radius of bowl* $= a$.

57, line 3, *for* Art. 39 (i) *read* Art. 35 (i).

— line 6, *for* $\dfrac{OR}{AQ-AO}$ *read* $\dfrac{OR}{Aq-AO}$.

— line 4 from foot, *for* AQ *read* Aq.

59, line 2 below fig., *for* fig. 38 *read* fig. 37; and in fig. insert (i) to the upper, (ii) to the lower portion.

65, fig. 40 (ii), *for* S *read* S'.

— — (iii), *for* S *read* S''.

66, (iii) line 2, *for* T *read* T'' and *for* N *read* V.

73, line 5, *for* 49 *read* 48.

— line 2 from foot, *for* 52 *read* 51.

83, line 12, *for* AO *read* AO_1.

— line 18, *for* O *read* o.

85, line 3, *for* $\dfrac{1}{f+CQ}$ *read* $\dfrac{f}{f+CQ}$.

90, line 10, *for* PCp_1 *read* PC_1p_1.

94, line 5, *for* fig. 53 *read* fig. 54.

95, line 6 from foot, *for* C_2 *read* C_1.

120, line 2, *for* SE *read* VE.

GEOMETRICAL OPTICS.

CHAPTER I.

INTRODUCTION.

There are certain ideas connected with light and vision which are familiar to us all from the moment that our powers of observation exist. We know, for instance, that some bodies send out light of their own, such as the sun, or a gas-flame, or a red-hot coal; whereas others send out only light which they have received from something else, as, for instance, the bright spots on the polished back of a chair which come from the light of the gas-flame, or the fire, or the moon, who sends us the light which she gets from the sun. Then again, we cannot but feel sure that light comes from the object to our eye in straight lines. As an illustration of this there is the fact that if a small round hole be made in a sheet of paper, and a steady candle-flame held before it, the light which passes through the hole will make an image of the flame upside down on another sheet of paper held parallel to the first. But this is only true so long as we suppose the light to be passing through the same "medium," as it is called, that is, for instance, so long as it is passing through the same kind of air. If in its course it come to a surface of water, the same thing would happen to it as happens to a ball passing through the air which is suddenly struck by another ball thrown from the side; *i.e.* its course is instantly changed from the straight line in which it *was* travelling to another straight line.

2. Before going any further we will give the definition of some terms which we shall often have to use:—

(i) Any point of a luminous body is sending out light in every direction in straight lines. It is therefore called a "*centre of light*," or a "*luminous point.*"

(ii) And any one of such straight lines is called a "*Ray.*"

(iii) Any number of consecutive Rays form a "*pencil.*" The most convenient form in which to imagine a pencil is a solid cone of rays, of which the centre of light is the vertex.

(iv) The middle ray, or axis of the cone of rays, is called the "*Axis*" of the pencil*.

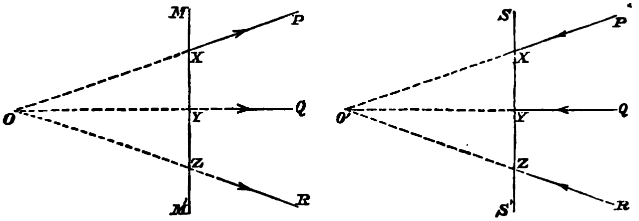
(v) When the rays forming a pencil of light come *from* a point, or *as if from a point*, the pencil is called a "*divergent*" pencil. When they come *to* a point, or *as if to* a point, it is called "*convergent.*"

---

* The ancients used to assume that the rays of a pencil were not close together, but separated like the fingers of the hand spread. By this they accounted for the fact that occasionally, in looking for a small object, like a needle, we cannot see it when it is under our nose.

Thus if MM' were a mirror, it is possible to see that rays might come off that mirror as if from O; or to fall on the screen SS' as if they were going to O'. Such pencils are respectively divergent and convergent, as much as if the rays really started from O or met at O'.

Fig. 1.



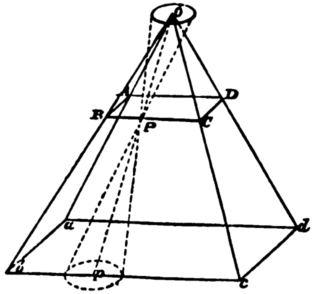
We shall often have occasion to talk about pencils of light falling upon plane surfaces and others: in all such cases, unless expressly mentioned, we shall suppose that the axis of the pencil falls perpendicularly on the surface.

(vi) Any substance through which light passes, such as air, water, &c., is called a "*medium*," and it is called "*uniform*," when its density, upon which depends the amount of bending that a ray suffers (Art. 2), is the same at every point. All media at present are to be considered uniform.

3. The following example may serve to familiarize the reader with the idea of a ray and a pencil of light. "Light emanating from a luminous circular disc placed in the ceiling of a room, passes through a rectangular aperture in the floor. Ascertain the form of the luminous patch on the floor of the room below."

(i) Let O be the centre of the disc; and let a straight line of light from O travel round the hole $ABCD$: let $abcd$ be the limit of light traced out by it upon the floor below.

Fig. 2.



(ii) Take any point P on the boundary of the hole, and think of all the rays which come from the disc and pass through P. They will make a circular patch at p on the boundary found in (i).

(iii) Let P travel all round $ABCD$: then the circular patch travels all round $abcd$, and radius of circular patch : radius of disc $= Pp : PO$.

If the hole be large compared with the disc, the travelling circle will be small, and the patch will therefore be similar in shape to that of the aperture, but with rounded corners.

If the disc be large compared with the hole, the general appearance will be a circle, whatever the shape of the aperture.

This example has an historical interest of its own. The first explanation was given by Maurolicus of Messina, in 1575, in words tantamount to our explanation. The following is translated from Monteucla's *History of Mathematics* (Part iii. Bk. v. 2):

"Why, asked Aristotle, does a ray of the sun, after passing through a hole of any shape, triangular for instance, or even a mere straight slit, if it is intercepted at a certain distance, always form a circle? and, still more strange, why, when the sun is partly eclipsed, does this ray in passing through the same hole as before, form an image precisely similar to the part of the sun's disc which is not yet hidden? This problem, which had been up to that time the despair of philosophers, had compelled them to say with Aristotle, that apparently the light took a certain curvature, or a similarity to the luminous body, which it regained immediately on freeing itself from the obstacle which had forced it from its shape......To explain this phenomenon, we will first notice, with Maurolicus, that each point of the aperture is the vertex of two opposite cones, one of which has the sun for its base, and the other, if cut by a plane perpendicular to its axis, will produce a luminous circle, the size of which varies as the plane of intersection is removed further from the opening. Thus there are depicted on the plane as many equal circles of light as there are points in the opening. And thus, if we draw on this plane a figure equal and similarly placed to that of the aperture, and at each of its points or only at those on its boundary describe a multitude of circles, the figure which they will form will be precisely that of the image of the sun, received at a distance proportional to the size of these circles. But in proportion as we increase the size of our circles, we shall see that the figure which results from them will approach nearer to a plain circle, and this is easy of demonstration........... But why does the sun, in a partial eclipse, form in a dark room the figure of a crescent, whatever is the shape of t opening? The explanation is the same as that just give If we have a figure of any shape on a plane, and if fro each of the points of its boundary we describe a successio of other figures similar to each other, and similarly placed, the resulting figure will approach nearer to that of those which are the greatest. If these are triangles, the whole figure will be a triangle; if a crescent, the figure formed will resemble a crescent."

CHAPTER II.

PLANE REFLEXION.

4. WHEN rays of light fall upon any surface, they are ither sent back from the surface, or else they enter the urface: in the first case they are said to be "*reflected*" rom the surface, in the latter case they are said to be '*refracted*" into the surface. At present we will only conider the laws and some of the phenomena of relexion.

Some surfaces, such as polished metal, do not refract t all: others, like water, partly reflect and partly refract, he amount reflected bearing a proportion to the amount efracted, depending on the angle at which the rays meet he surface. Thus, if a number of rays fall perpendicularly n to the water they are nearly all refracted: if they fall at n angle to the water, some of them are reflected, and the nore their direction slopes to the water the more rays are eflected, until at last none are refracted at all.

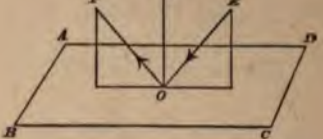
5. Suppose a single ray falls upon any reflecting surice; then

(i) The incident and reflected rays lie in one and the same plane with the normal to the surface at the point of incidence, and on opposite sides of it.

(ii) The angles which the incident and reflected rays make with the normal are equal to one another.

Thus, if OP be a normal to $ABCD$ at O, and if a ray of light EO fall upon the surface at O, and is reflected in the direction OF, then EO, OP, and OF are all in the same plane, and the angle FOP = angle EOP.

Fig. 3.



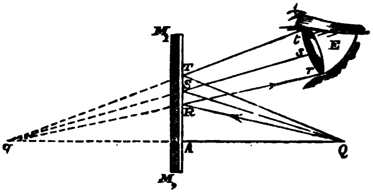
Thus the action of a reflecting surface upon a ray of light is precisely the same as that of a wall upon a very elastic ball thrown obliquely against it. In playing racquets, for instance, one naturally, without thinking of it, places oneself so as to receive the ball after coming off the wall at about the same angle as it went on; and the more elastic the ball is, the more nearly is this true; and if the ball is very elastic indeed, it is very nearly true; and when instead of a ball we think of a ray of light, which is quite elastic, this is absolutely true.

From these two properties, which were known by the ancients as well as by ourselves, all the phenomena of reflexion can be explained. We will consider a few of the simplest—at present supposing the reflecting surface to be quite plane.

6. The simplest and at the same time the most common instance of plane reflexion is that of a looking-glass; in which we appear to see the image of any object, placed in front of the mirror, just as far behind the mirror as the object itself is in front.

Let M_1M_2 be a mirror, edgeways to us. Let Q be a point of an object in front of the mirror: Q is sending out rays in all directions. E is an eye looking into the mirror. Draw QA perpendicular to the mirror. Let any ray QR strike the mirror at R: it will be reflected back in the line Rr, where $\angle M_1Rr = M_2RQ$. Produce Rr backwards through the mirror, to meet QA produced in q. Then the angle $M_1Rr =$ angle M_2RQ and also $=$ angle qRA; $\therefore qRA = M_2RQ$, and the right angle $qAR = RAQ$, and the side RA is common. Therefore, by Euclid I. 26, the other sides, &c. of the triangles are equal; $\therefore qA = QA$.

Fig. 4.



So if we take any other rays QS, QT... they will be reflected as Ss, Tt, and these lines when produced backwards will all pass through q.

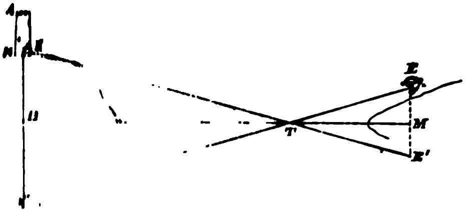
Now another way of saying this is to say that the eye at E will be affected by these rays in just the same way as if they all started from q. That is the eye will see an image of Q at q, just as far behind the mirror as Q is in front, since $Aq = AQ$.

In this we have taken only a small piece of the divergent pencil from Q, of which QA is the axis. After striking the mirror it becomes a divergent pencil from q, according to our definition of a divergent pencil in Art. 3 (v), for the rays meet the eye as if they came from one point.

An image through which the rays do not really pass, as q, is called "*virtual.*" Those through which the rays really pass are called "*virtual*." Thus all the images formed by plane reflexion are virtual. It should be noticed that if we are considering the rays after reflexion, we may practically consider Q and the mirror to be entirely removed, and the point q to be a real point sending out luminous rays.

Suppose that AB was a tower on some high ground on one side of a piece of water, and that E was an eye on the other side looking into the water: will the eye see the whole of the tower by reflexion?

Fig. 5.



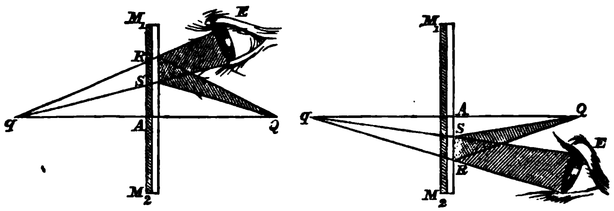
The eye at E will clearly see just as much as an eye at E' where $ME' = ME$.

7. If it is wished to trace the course of the rays which come from the object to the eye, before and after reflexion, we can at once obtain the construction from Fig. 4. It is this. "Join the image with the eye, by straight lines, and shade between these lines from the mirror up to the eye. From the points where these lines cut the mirror, draw straight lines to the original point, and shade. The shaded part will represent the required rays."

Thus, using the same letters as before, join q the image with the eye E: and shade from R and S to the eye. Join R and S to Q and shade. The shaded part represents the rays which fall from Q on the mirror, and thence come to the eye.

Whenever, in future investigations, we require to trace the course of a pencil of rays from an object to the eye, after any number of reflexions, we shall make use of an extension of this rule—viz.:

Fig. 6.



(1) Join the eye to last image, and shade to the eye from the points where these lines cut the last surface.

(2) Join these points to last image but one, and shade from them up to last surface but one.

(3) Join the points last found to last image but two, and shade from them up to last surface but two, and proceed in this manner, until we come to the first surface, when we join the points found in it to the object, and shade as before.

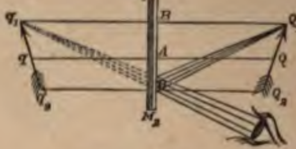
We thus, in fact, trace the rays back from the eye to the object.

8. The result of Article 6 applies, of course, to any point of an object. Thus every point of an object has an image of itself formed just as far behind the mirror as the point itself is in front of the mirror.

Thus, if Q_1QQ_2 be any object placed in any position in front of the mirror M_1M_2, it will have an image of itself q_1qq_2 where

$$Aq = AQ,$$
$$Bq_1 = BQ_1,$$
$$Cq_2 = CQ_2.$$

Fig. 7.

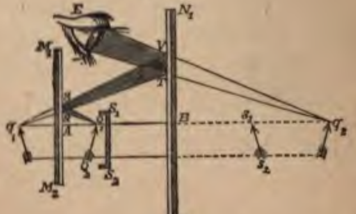


The images q_1, q, and q_2 all being virtual.

9. Let M_1M_2 be a mirror, Q_1Q_2 any object in fron it, the image of which is at q_1.

Let N_1N_2 be another mirror, exactly facing the for but let there be a screen S_1S_2 so placed that no rays come from Q_1Q_2 on to N_1N_2; *i.e.* so that no image of can be formed by direct reflexion at N_1N_2.

Fig. 8.



Consider any point Q_1. Rays of light Q_1R, Q_1S, c from Q_1 on to the mirror M_1M_2 and are reflected from if they came from q_1, where $Aq_1 = AQ_1$. They then fal N_1N_2 as RT, SV. Now since they may be considere rays diverging *from the point* q_1, SV and RT will of course form an image of q_1 at q_2, where $Bq_2 = Bq_1$, which will be seen by an eye looking into N_1N_2.

Thus the rays which reach the eye come as if from q_2, *i.e.* the eye E sees an image q_2 of Q_1. And similarly for all the other points of Q_1Q_2.

The shaded part represents the rays traced from Q_1 to the eye E, by means of the rule in Art. 8.

We can easily find the distance of q_2 from either mirror.

For $Aq_1 = AQ_1$; $\therefore\ q_1B = 2AQ_1 + Q_1B$;

$$\begin{aligned} \therefore \text{ also } q_2B &= 2AQ_1 + Q_1B \\ &= 2AQ_1 + AB - AQ_1 \\ &= AQ_1 + AB; \\ \therefore\ q_2A &= AQ_1 + 2\,.\,AB. \end{aligned}$$

10. If the rays were again allowed to fall on the mirror M_1M_2 before coming to an eye, the eye would see an image q_3 behind the mirror M_1M_2, such that

$$Aq_3 = Aq_2 = AQ_1 + 2AB.$$

If again they fell upon N_1N_2 before coming to the eye, a fourth image q_4 would be formed as far behind N_1N_2 as q_3 is in front; whence $q_4A = AQ_1 + 4AB$, and so on.

In this investigation we have carefully considered only a pencil of rays which fall *first* on the mirror M_1M_2, in order to save confusion: but it is easy to see that in precisely the same manner there will be a series of images formed by a pencil of rays which fall first on N_1N_2, of which s_1s_2 would be the first.

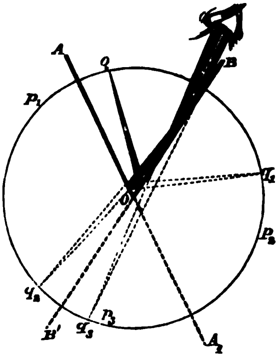
It is perhaps necessary to remark that difficulty may be felt at the phrase "the image *formed* by reflexion." It will be felt that this is a wrong word to use, since the rays do not *actually* pass through q_1, q_2, &c. Of course there is no image formed—but if an eye is placed so as to catch t rays which come as if from one of the points q_1, q_2, &c. a image is *practically* formed there.

Explanation of the Kaleidoscope.

11. (1) An object is placed between two mirrors inclined to one another at any angle. To find the number and positions of the images.

Let OA and OB be the mirrors. Let Q be the object, placed between them. Let us consider only light falling first on OB.

Fig. 9.



Then, of course, Q has an image q_1, where q_1 is as far behind OB as Q is in front, *i.e.* after being reflected at OB the rays will appear to diverge from q_1 .... (1).

Next, these rays will now fall on OA, and, since they practically come from q_1, they will, after reflexion at AO, appear to diverge from q_2, where q_2 is as far behind OA as q_1 is in front of it.. (2).

Lastly, these rays will now fall on OB, and, since they practically come from q_2, they will, after reflexion at OB, appear to diverge from q_3, where q_3 is as far behind OB as q_2 is in front of it.

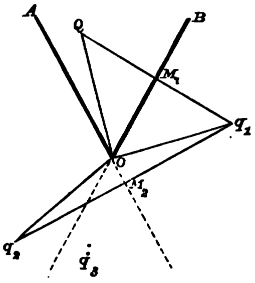
We say *lastly*, because it is clear that no rays which come off OB as if from q_3 can possibly fall on OA, and there will therefore be no more images.

The figure shews the course of a small pencil which comes to the eye after suffering the greatest number of reflexions possible. On entering the eye the rays seem to diverge from q_3, *i.e.* the eye sees an image there.

The eye will, of course, see *all* the images, although we have traced only the course of the pencil by which it sees the last one. Rays, for instance, will fall from Q on OB and be reflected directly to the eye, which will thus see q_1. Similarly rays will come to the eye after two reflexions, and the eye will therefore see q_2. The student should draw figures for himself of these pencils, using the rules given in Article 8.

(2) We will now shew that all these images lie upon a circle whose centre is O and radius OQ.

Fig. 10.



For $QM_1 = q_1M_1$ and M_1O is common; ∴ QM_1, $M_1O = q_1M_1$, M_1O, each to each, and the angle QM_1O = angle q_1M_1O; ∴ the base QO = the base q_1O.

In just the same way, since $q_1M_2 = q_2M_2$ and OM_2 is common, and the angle q_1M_2O = angle q_2M_2O, therefore $q_1O = q_2O$.

Thus $q_3O = q_2O = q_1O$, that is the points q_1 q_2 q_3 all lie on a circle, centre O, and radius OQ.

It must be observed that hitherto we have only considered rays which fall upon OB first. If we now consider those which fall first on OA we shall obtain three more images P_1 P_2 P_3. It can be shewn that if the mirrors are inclined to one another at any submultiple of two right angles, 30°, 60°, for example, the last images of the two systems of rays will coincide. Thus if our mirrors are inclined at 60°, the eye will see five images and not six (see figure in next article).

12. The positions of these images can be formed generally, supposing that we know the angles QOA and QOB.

For
$$QOq_1 = 2 \,.\, QOB \text{ .... (i)};$$
$$\therefore q_1OA = 2 \,.\, QOB + QOA;$$
$$\therefore q_2OA = 2 \,.\, QOB + QOA;$$
$$\therefore QOq_2 = 2 \,.\, QOB + 2 \,.\, QOA \text{ ...... (ii)},$$

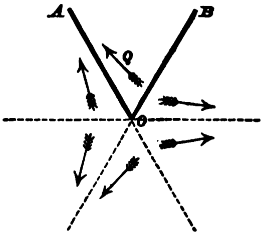
similarly
$$QOq_3 = 2 \,.\, QOB + 2 \,.\, QOA + 2 \,.\, QOB \text{ ...... (iii)},$$
$$QOq_4 = 2 \,.\, QOB + 2 \,.\, QOA + 2 \,.\, QOB + 2 \,.\, QOA,$$

so that the angular distances of $q_1\, q_2\, q_3\, q_4 \ldots$ from OQ are found each from the one before it, by adding alternately $2 \,.\, QOA$ and $2 \,.\, QOB$.

The figures which we have drawn are of course sections of the mirrors perpendicular to the line in which they meet. Although these images exist for an eye in the same plane as the section, yet in order that they might be seen by an eye out of the plane, the mirrors would have to be prolonged, so that rays, after striking them, might meet the eye.

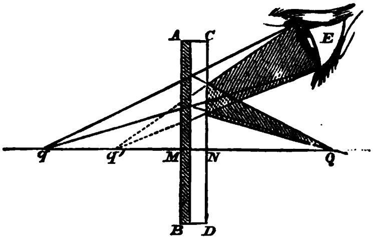
So that a virtual image may be described as an image that does not exist until there is an eye to receive the rays. The eye calls the image, though not the rays, into existence.

Fig. 11.



13. We can explain the reason why, when looking into a glass mirror, we very often see a double image of an object, one image being very faint.

Fig. 12.



Let *ABCD* be the mirror. Let *QNM* be perpendicular to the mirror. *AB* is the silvered back.

The strong image is formed by the light which, passing into the glass, is reflected from AB. But a very little of this light never gets to AB through CD at all, but is reflected from CD. Thus we obtain another very faint image q', where $q'q = 2MN$.

For

$$\begin{aligned} Mq &= MQ = MN + NQ, \\ Nq' &= NQ;\ \therefore Mq' = NQ - MN; \\ \therefore qq', \text{ which} &= Mq' - Nq' = 2MN, \\ \textit{i.e. } qq' &= \text{twice thickness of the plate.} \end{aligned}$$

But at present in talking of glass mirrors we shall always neglect the thickness of the plate. The shaded portion of this figure is intended to represent the rays forming the faint image.

Note. We have here, for convenience of explanation, neglected the refraction which the light suffers on entering, and emerging from, the glass.

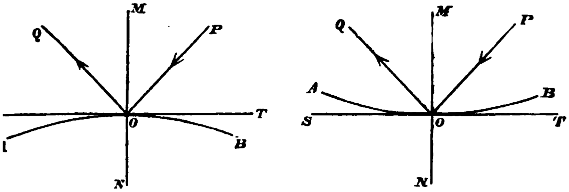
CHAPTER III.

REFLEXION AT A CURVED SURFACE.

ı. We have hitherto thought only of the images d by rays which fall upon and are reflected from plane rs, and have seen that in all such cases we shall have age of any object just as far behind the mirror as the t itself is in front; and the only rule which we used to :his was that after a ray of light struck the mirror it off it again, making the same angle with the mirror or ı line perpendicular to the wall as it did on striking it.

ow precisely the same thing happens to a ray when it s a curved surface; but the result is not the same, ıe image is not seen just as far behind as the object is nt.

Fig. 13.



'hus let *AOB* be the section of any curved surface, : by the plane of the paper, and let a ray from *P* strike ıurface at *O*: draw a plane of which the section is

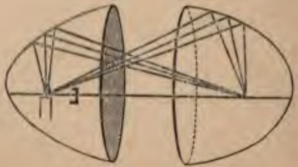
ST touching the curved surface in O, and let MON be a line perpendicular to ST: then if OQ be the ray after reflection $Q\hat{O}M = P\hat{O}M$; and the direction of OQ is therefore just the same as if the ray had struck the plane ST instead of the curved surface.

(It will be very useful sometimes when we have a ray striking a curved surface, to treat it as if it struck the plane surface which touches the curved surface at the point.)

15. (i) The simplest case of a curved surface is *sphere.* Suppose a light placed in the centre of sphere. A line from the centre to any poi on the circumference is perpendicular to, that makes equal angles with, the tangent at that poi But the incident and reflected rays also made eq angles with the tangent, whence it follows that t reflected ray returns along the incident ray, *i.e.* the reflected rays pass through the centre.

(ii) Take an *elliptical* surface with a light in one foc and suppose a ray to fall on the reflecting surfa Now we know by Optics that the incident and

Fig. 14.



flected rays make equal angles with the tang Also we know by Geometry that the lines from foci make equal angles with the tangent; whe it follows that a ray incident from one focus is reflected to the other—and similarly for all the rays.

An interesting experimental proof is the following. If a candle flame be placed in one focus and a piece of phosphorus in the other, the concentrated reflected rays of heat, which follow the same law as light as regards reflexion, will ignite the phosphorus, although the direct light of the candle be kept away by a screen.

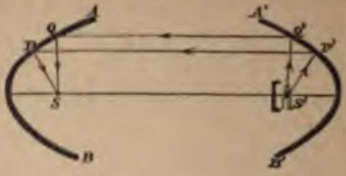
(iii) Consider a *parabolic* mirror with a light in the focus. By precisely similar reasoning it may be seen that all the reflected rays will go off in a beam parallel to the axis, the breadth of the beam of course depending on the size of the parabola.

Upon this principle the construction of one kind of lighthouse is based. A mirror very nearly in the form of a paraboloid of revolution, *i.e.* the surface formed by a parabola revolving about its axis, has a very strong light in the focus. The rays, therefore, after reflexion go off in a very strong beam of rays very nearly parallel. When this beam has travelled a long distance, its rays, though still much condensed, have spread sufficiently to form a broad beam of strong light: when a ship passes through this beam, it sees the light at the lighthouse, which it would never do, if there was merely the light burning and no mirror behind to reflect its rays.

It should be noticed that, just as light from the focus on reflexion goes off parallel to the axis, so light entering parallel to the axis would, after reflexion, make a beam converging to the focus. So that if we wish to collect a number of rays from the Sun for instance on to one spot, we can do it by placing a parabolic mirror behind the spot, the larger the better, having the spot for focus.

Suppose that we have, as before, a candle in S', and a piece of phosphorus in S; but that the direct heat from S' is kept off by a screen. Place a parabolic mirror $A'B'$ behind S', with its focus in S' and its axis in the line SS'. Place another parabolic mirror AB with its focus in S and its axis in SS'. The rays from S' will come off $A'B'$ and on to AB parallel to the axis: they will therefore converge to S, and will light the phosphorus.

Fig. 15.



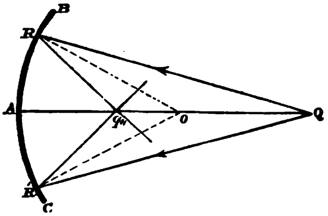
16. A spherical surface is the only one of those which we have named which is easily made, and is constantly used. We will therefore now consider the phenomena connected with reflexion at a spherical surface.

In doing this it will be found better to consider a pencil of rays as a series of conical shells of rays, one within the other, all the vertices coinciding at the luminous point.

Let Q be a luminous point inside the sphere: let O be the centre of the sphere, and let QO produced meet the surface in A. Let the figure be a section made by the plane of the paper, in which both Q and O are supposed to lie.

(i) Consider now a pencil of light of which QOA, which falls perpendicularly on the mirror, is the middle line or axis: then, since QOA is the only ray of the pencil which falls perpendicularly, it is the only ray which will be reflected off perpendicularly,

Fig. 16.



and all the other rays will be reflected symmetrically round it, and it will therefore be the axis of the reflected pencil.

(ii) Of that pencil consider only one of the conical shells of rays, viz. that whose vertical angle is RQR'.

(iii) Of that shell consider only the ray QR.

Join OR: then since BAC is part of a circle RO is the ormal at R. Therefore the ray QR is reflected back in irection Rq'', where angle ORq'' = angle ORQ;

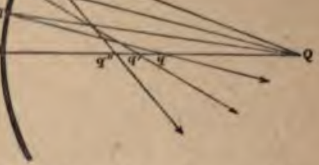
$$\therefore \text{ by (Euc. VI. 3)} \frac{QR}{Rq''} = \frac{QO}{Oq''} \cdots\cdots\cdots \text{(i)}.$$

Similarly QR' will be reflected to q'', and so will every ay of the shell whose vertical angle is RQR'. So that the *ivergent* conical shell from Q forms, after reflexion, a *con-ergent* conical shell to q''. The point q'' is called the *Focus*" of the shell.

Now consider another conical shell whose vertical angle ; SQS'. This will in like manner have a focus q' near to ", and thus each shell will have its own particular focus on he line QA.

17. It can be proved that the nearer to A that the rays from Q fall, *i.e.* the smaller we take the vertical angle of our shell, the more nearly do the foci coincide. That is, qq' is less than $q'q''$; $q'q''$ is less than $q''q'''$, and so on.

Fig. 17.



The point where the reflected rays of the shell with the least imaginable angle cut the axis is called the "*Geometrical Focus*" of the pencil. Close to it then the points $q'q''$ &c. lie thickest, that is, more reflected rays pass close by that point. That is, an eye looking into the mirror would see the strongest light if it happened to be placed so as to catch the rays which before reflexion fall on the mirror very close to the axis. This point is therefore usually taken for the image of Q. If we agree to think only of rays which fall not far away from the axis we can say that, *approximately*, after reflexion they will converge to the point called the "Geometrical Focus," and an eye will therefore see an image of Q at that point. This image then is different from that formed by plane reflexion, in that it is *real*, whereas in plane reflexion it was virtual. But it is not *distinct*, since the rays by which it is seen do not pass accurately through a point, as they appeared to do in plane reflexion.

18. If the student will apply precisely the same reasoning to the case of a convex mirror, he will see that precisely

the same results follow, with this exception—the image will be *virtual* and not real.

The following figures will serve to shew the course of rays to the eye.

Fig. 18.

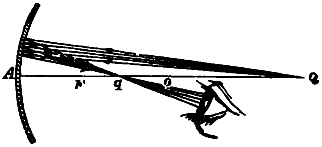
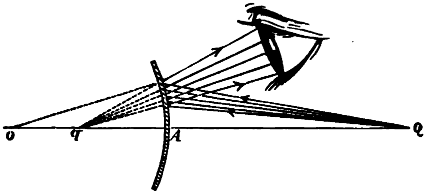


Fig. 19.



19. *To find the position of the "Geometrical Focus" for the concave mirror.*

Since the Geometrical Focus is a point of such importance, it is of great consequence to know exactly where it lies relatively to the object.

In Art. 16 it was proved that $\frac{QR}{Rq} = \frac{QO}{Oq}$, where QR is any ray from Q.

Suppose now that QR is taken close to the axis. Then QR may be regarded as equal to QA; since by bringing the ray closer and closer to the axis, the difference between QA and QR may be made as small as we please. Similarly Rq becomes Aq, where q is *now* the G. F.

Thus

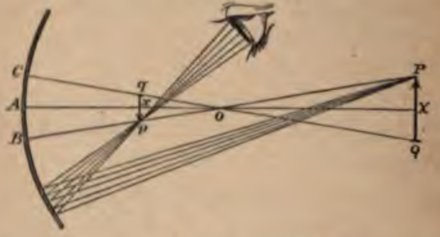
$$\frac{QA}{Aq}=\frac{QO}{Oq} \text{ or } \frac{QA}{Aq}=\frac{QO}{(AO-Aq)}; \quad \therefore \quad \frac{QA}{Aq}=\frac{AQ-AO}{AO-Aq},$$

$$\text{whence } \frac{1}{AQ}+\frac{1}{Aq}=\frac{2}{AO}.$$

Proceeding in a similar way for the convex mirror it would be found that $\frac{1}{AQ}+\frac{1}{Aq}=-\frac{2}{AO}$.

20. Thus finally the image of any point of an obje lies on the normal to the mirror from that point, at distan Aq from the point where this normal cuts the mirror, su that $\frac{1}{AQ}+\frac{1}{Aq}=\frac{2}{AO}$.

Fig. 20.



This, as we say, applies for any and every point of t object. Thus an image of any object PXQ will be form smaller, and inverted.

Thus, in the figure, $\frac{1}{AX}+\frac{1}{Ax}=\frac{2}{AO}$,

$$\frac{1}{BP}+\frac{1}{Bp}=\frac{2}{AO},$$

$$\frac{1}{CQ}+\frac{1}{Cq}=\frac{2}{AO}.$$

Similarly it will be seen that in the convex mirror the image is smaller and erect.

21. The Geometrical Focus of a pencil of rays from an infinite distance—*i.e.* a pencil of parallel rays, is called the "*Principal* Focus" of the mirror.

Now when AQ is infinitely large, $\frac{1}{AQ}$ is indefinitely small and may be left out, and therefore the formula $\frac{1}{AQ}+\frac{1}{Aq}=\frac{2}{AO}$ becomes $\frac{1}{AF}=\frac{2}{AO}$, where F is the principal focus. That is, F is midway between A and O, or the image of a star would be seen there, since a star is a very long way off.

22. There is now one point to notice to which we shall often again refer, regarding the *signs* of lines. We will always make use of the following convention. All lines measured in the direction *opposite* to that of the incident light are to be regarded as positive lines, and *vice versa.* Thus in Fig. 16 AO is positive, and therefore that is taken as the standard case. AQ must, of course, always be positive for divergent pencils. Now consider the case of a *convex* mirror with a divergent pencil. AO is then measured in the same direction as the incident light and is therefore negative. The formula, therefore, which for the concave mirror was $\frac{1}{Aq}+\frac{1}{AQ}=\frac{2}{AO}$ will now be $\frac{1}{Aq}+\frac{1}{AQ}=-\frac{2}{AO}$. (Compare Art. 47.)

This point will present no difficulty to those readers who are familiar with the elements of Analytical Geometry.

Suppose now that we wish to find the position of the *Principal Focus.* This is the Geometrical Focus of a pencil of rays from an infinite distance. Thus if F is the Principal Focus, $\frac{1}{AF}+\frac{1}{AQ}=\frac{2}{AO}$. But $\frac{1}{AQ}=0$, since $AQ=\infty$; $\therefore \frac{1}{AF}=\frac{2}{AO}$; *i.e.* the Principal Focus lies halfway between the mirror and its centre—or the light from a star would make a bright spot there.

23. Thus, finally, we have the following facts connecting Q and q.

(1) Q and q lie on the same normal. For all rays from Q when reflected pass through q, and since the normal ray is reflected along itself, q must lie upon it.

(2) If A be the extremity of the axis, and the pencil be divergent, $\frac{1}{Aq}+\frac{1}{AQ}=\pm\frac{2}{AO}$, according as the mirror is concave or convex.

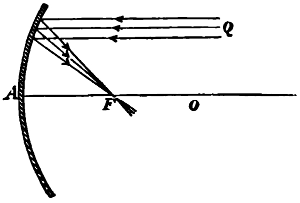
(3) Since $\frac{2}{AO}$ = constant quantity, as AQ increases or decreases, Aq does just the opposite, in concave mirrors; *i.e.* Q and q move in opposite directions. In the convex mirror they move in the same direction.

(4) Since, in the concave mirror, when Q goes farther from A, q comes nearer, and, since, when Q is at its farthest, q comes to F the principal focus, therefore Q and q must lie on the same side of F always. Similar reasoning proves the same fact for the convex mirror.

By a principle which we have used before (Art. 15), if light can travel from any luminous point Q to any other point q, by any number of reflexions, then if a light be placed at q its rays will go to Q by precisely the same path that it used before. This would be the case in fig. 16. Or we can see it at once from the equation $\frac{1}{Aq} + \frac{1}{AQ} = \frac{2}{AO}$, where it makes no difference if we interchange AQ and Aq.

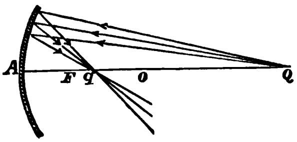
24. We will investigate the motion of Q and q still further.

Fig. 21. (i)



(1) Let F be the principal focus of the concave mirror. Then rays from a point very far from A, that is parallel to the axis, converge very nearly to F (fig. i).

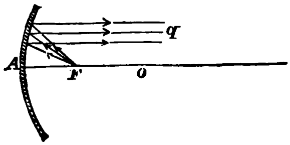
(ii)



(2) Let Q come nearer the mirror from the right. The rays from Q then converge very nearly to some point q to the right of F, since if AQ decrease, Aq must increase (fig. ii).

(3) Let Q come to O the centre. The reflected rays all converge to O. Q and q meet (for if $AQ = AO$ so does Aq).

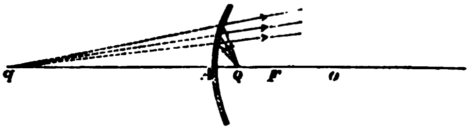
(iii)



(4) As Q goes to the left, Q and q cross, and q goes off to the right, until, when Q has come to F, q has gone to an infinite distance (fig. iii).

(5) As Q still approaches A, q as it were turns the corner at infinity and appears again to the left of A, approaching A (fig. iv).

(iv)



(6) When Q arrives at A, q does so also; for from $\frac{1}{Aq}+\frac{1}{AQ}=\frac{2}{AO}$, if $AQ=0$ so must Aq.

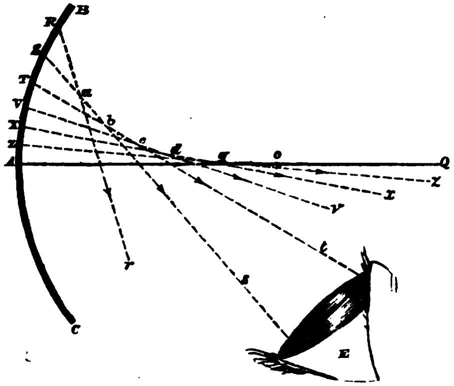
CHAPTER IV.

CAUSTIC CURVES, &c.

25. It cannot be too often recollected that the point *q* is not the point through which *all* the reflected rays pass, for there is no such point. But close to it the rays which fall on the mirror very close indeed to the axis, from *Q*, cut the axis after reflexion. If we take rays from *Q* which fall on the mirror further from the axis, we find that they cut the axis at appreciable distances from *q*; and also that the points of section of the axis become more and more scattered as the rays fall on the mirror further and further from the axis, all being on the side of *q* towards *A*.

Thus, if the dotted lines represent the reflected rays,

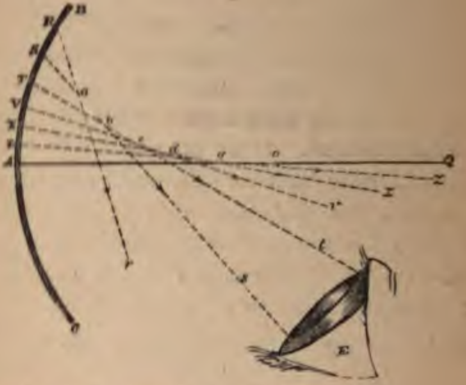
Fig. 22.



the rays before reflexion not being drawn, to save confusion in the figure, fig. 22 will represent the course after reflexion of the rays forming the part of the pencil which strike the mirror above the axis.

The figure, as before, represents a section of the spherical mirror and pencil by the plane of the paper.

Fig. 22.



Let *Rr*, *Ss*, *Tt*... be the reflected rays.

Let *Rr*, *Ss* intersect in *a*,

Ss, *Tt* .... *b*,

Tt, *Vv* .... *c*, and so on.

Now suppose an eye at *E* receives the rays *Ss*, *Tt*. It receives them as if they both came from *b*, since they cross one another there. That is, *the eye sees a bright spot there;* brighter than elsewhere along either line, since double light comes from that one spot.

Similarly, if the eye received *Rr*, *Ss*, and *Tt*, it would see two bright spots *a* and *b*: for it would receive *Rr* and *Ss* as if from *a*, and *Ss*, *Tt* as if from *b*. And if it received *Rr*, *Ss*, *Tt*, *Vv*, it would see three bright spots *a*, *b*, *c*; and so on.

And if we conceive a great number of rays falling upon and being reflected from the mirror, it is plain that there will lie a great number of spots of light very close together. All these points form a curve, whose cusp or stopping-point is the geometrical focus. Each consecutive pair of rays comes to the eye as if from some one point on this curve; that is, the eye sees points on the curve *abcdq*. Since the pupil of the eye is a very small thing, only a few consecutive rays can enter it, and therefore it can only see a very small piece of the curve at a time.

This curve is called the *caustic* curve.

If we caused *BAC* to revolve about the axis we should obtain a portion of a spherical mirror, but as *BAC* revolved, so would *abcdq*, and therefore, instead of a caustic curve, we should have a caustic *surface*.

26. Figure 23 is intended to represent the caustic curve formed by reflexion from a semicircular mirror of a pencil of rays which fall on it parallel to the axis. The cusp will be at *F*, midway between *O* and *A*; since that is the position of the principal focus. The curve on the paper in the experiment suggested will be brightest at and close to *F*, since the points of intersection are closer there than elsewhere. Or, supposing the rays to come directly to the eye, an eye at *E* for instance will see stronger light than an eye at *E′*, since the rays when they come to *E′* are more spread than those which come to *E*, and therefore fewer of them enter the eye.

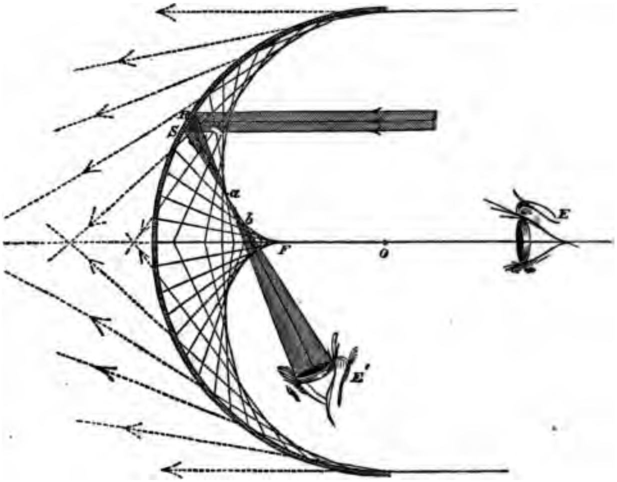
We have hitherto considered the geometrical focus as the image of a luminous point; but that was only when we supposed the eye on or very close to the axis, when the rays passing through the geometrical focus come to the eye. But, clearly, for an eye at E' the image which it sees is not at the geometrical focus but at that point on the caustic curve through which and close to which the rays coming to the eye pass. Thus, if the light of a star was shining on the circular mirror, an eye at E' would see a blurry image of the star at ab.

In some cases the caustic curve reduces to only one point: the caustic of a *circle*, the luminous point being in the centre, is the centre. The caustic of an *ellipse*, the luminous point being in one focus, is the other focus; of a *parabola*, for rays coming parallel to the axis from a great distance, it is the focus. The caustic of a plane surface is the point as far behind the surface as the light is in front, on the same normal. In fact it is always a single point when the reflected rays pass or seem to pass *accurately* through it, as is true in all these cases.

27. Since the rays $Rr, Ss, Tt\ldots$ (Art. 25) actually pass through the points $a, b, c\ldots$ the caustic curve exists, although no eye receives the rays. This can be shewn very clearly thus. Bend a piece of watch-spring into an arc of a circle, and fix it edgeways on a sheet of paper. Then place a light in the axis, the least bit above the plane of the paper; or, if the sun is shining, let the rays of the sun fall on it. The reflected rays will fall on the paper and make a brilliant curve, the whole of which is visible to the eye, since it is seen by the light *on the paper*, not by the directly reflected rays. A rough caustic curve may be observed on the bottom

tea-cup, formed by the rays which have been reflected the sides.

Fig. 23.

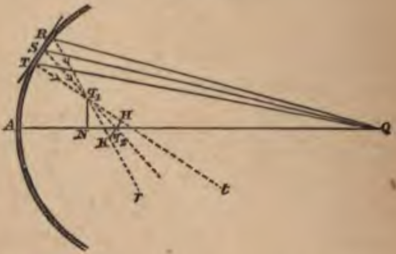


CHAPTER V.

Focal Lines.

28. We have shewn in the last chapter that when pencil of light is reflected from a spherical surface, th reflected rays of light all touch a certain surface called th caustic surface; or, if we consider only the rays in the pla of the paper, they all touch a curve called the caus curve, whose cusp is at the geometrical focus of the pen For the present we will only consider rays in this plane.

Fig. 24 (i).

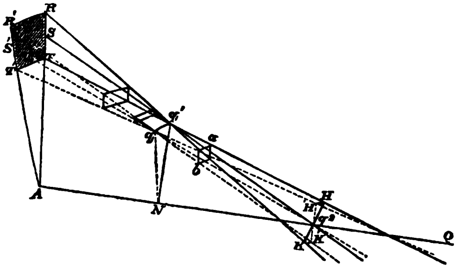


If now we consider only a very few of these rays, a close together, they will, after reflection, touch the causti curve at points very close together, and if we take ther close enough together, we may consider them as passin ıgh only one point on the caustic, but cutting the axis ıe direct pencil from Q, of which they form a part, in ̄ent points.

̲et then QR, QS, QT be rays from Q in the plane of ›aper falling on the mirror so close together that after ꞡ reflected they may be supposed to go very nearly ıgh one point q_1 on the caustic, and cutting the axis of the pencil of which they form a very small part in rent points very close together. Let q_2 be the point ̄e the middle ray cuts QA; then all the other rays :h fall on the mirror between R and T, pass through line HK, which we will suppose drawn parallel to the ;ent at S.

ꝛ9. Now suppose the whole figure to rotate through a *small angle* about the axis AQ, so as to form a solid re.

Fig. 24 (ii).



Then (i) RT describes a portion of a ring, the shaded part in the figure representing the part on which the rays fall.

(ii) The point q_1 describes a very small arc of a circle, q_1q_1', radius Nq_1, so small that it may be regarded as a very small straight line perpendicular to the plane of the paper; and rays pass all through this small line.

(iii) HK, since it rotates about the point q_2, describes a figure which may be called an elongated figure of eight, so that there are rays passing through all the surface of the two triangles Hq_2H', Kq_2K'. But this figure of eight is so very nearly a straight line, since the angle through which we have turned the figure is so very small, that we may regard it as really such.

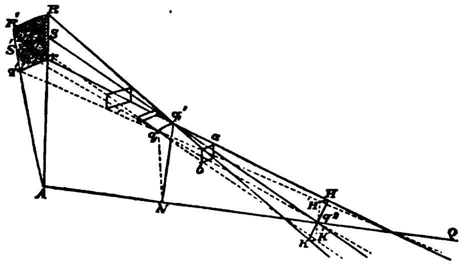
The line q_1q_1' is clearly at right angles to the plane RAQ, since it is formed by the rotation of the whole figure about AQ, which is in that plane.

Thus we may say, approximately, that all our rays this very small pencil after reflection pass through two ve small lines at right angles to one another. The first these, the one formed by the revolution of q_1, is called th "*Primary Focal Line*," the second is called "*The Secondar Focal Line*."

(One difficulty that the reader will probably have felt i this subject is the fact that in order that our results may b even approximately true, we are obliged to take only such very small pencils of light. But if he will consider what very small thing the pupil of his eye is, and what a smal pencil of light it can receive, this difficulty will probably vanish.)

30. If a section be taken of the reflected pencil at q_1 by a plane parallel to the tangent plane at S, we have seen that

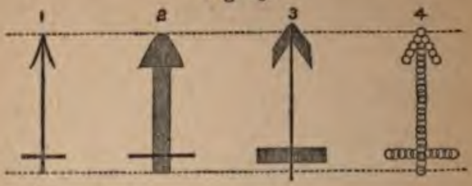
Fig. 24 (ii).



the section is approximately a straight line in that plane *perpendicular to* the plane RAQ. If this plane be supposed to move gradually from q_1 to q_2, remaining parallel to itself, the height of the section increases and the breadth diminishes in the moving plane, until at q_2 the section is a straight line *in* the moving plane, but at right angles to its former direction. Since this change goes on gradually, there must be one position of the plane at which the height and breadth of the sections must be the same, and the section, or the spot of light formed on the plane, is very nearly square. The section of the reflected pencil at this point, *i.e.* ab, may be called the "Section of least confusion" of the pencil, and if the corners be cut off $RTT'R'$, so as to make it oval, this section becomes nearly a circle, which is called the "Circle of least confusion."

Thus it can now be seen that if a screen were placed at q_1 an image of an arrow (fig. 25, 1) placed at Q would be seen thus (fig. 25, 2): while, if placed at the point q_2 the screen would receive an image thus (fig. 25, 3). And, lastly, if placed at the point where the circle of least confusion is formed, it would be seen thus (fig. 25, 4).

Fig. 25.



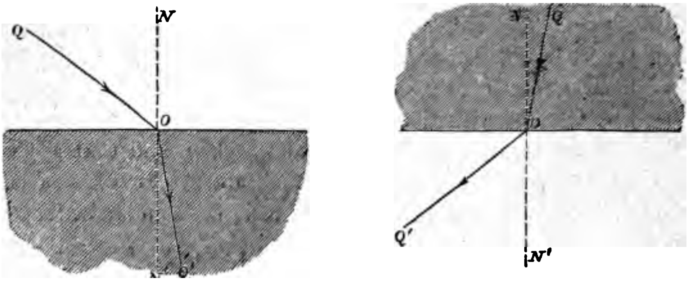
31. Thus, finally, an eye *on the axis* receives a small pencil which may be regarded as diverging approximately from a *single* point called the Geometrical Focus—since for a very small pencil *close to the axis* these two small lines and the circle of least confusion all come together. But supposing the eye not to be in the axis, and therefore not capable of receiving a small pencil which falls close to the axis, we see that we have found two ways of expressing the nature of the image which it sees of the luminous point, viz. either we may regard the pencil which enters the eye as diverging from a point on the caustic surface, or, more accurately still, from two very small lines at right angles to each other. Since the circle of least confusion is the nearest approach to a point that the reflected pencil can reach, we may take *it* as the image of Q, and if Q is of finite size, the image will be as in fig. 25 (4). The size of this circle clearly measures the indistinctness of the image.

CHAPTER VI.

Refraction at a plane surface.

32. (i) When a ray of light has been passing through any one medium of uniform density, and suddenly reaches another medium of different uniform density, it is found that it does not go on in its original line, but is bent aside on entering the new medium. It is also found by experiment, that, if it enters a dense medium from a rare medium, it is turned nearer to the normal, as in (i); if from a dense medium into a rare medium, it is turned further from the normal as in (ii).

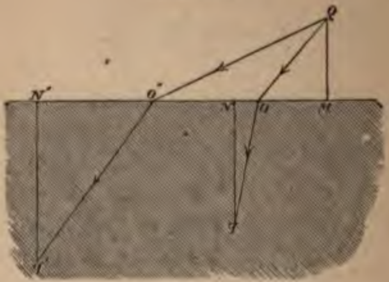
Fig. 26.



(i) (ii)

(ii) It was also found that the following relation held between the directions of the ray before and after refraction, Let QOq be a ray in the plane of the paper. Take $Oq = OQ$ and draw QM, qN perpendicular to the surface. Then the relation that holds is this. $OM : ON$ in a constant ratio,

Fig. 27.



whichever ray we take, supposing the media to remain unaltered. Thus, if $QO'q'$ be another ray, then $O'M : O'N'$ in the same ratio as before: if $OM = \frac{3}{2} ON$, then $O'M = \frac{3}{2} O'N'$ and so on.

33. The fact stated in Art. 32 (i) was first distinctly stated by an Arabian mathematician, named Alhazen, who lived about 1100; although the phenomena of refraction had been remarked sooner: Seneca, for instance, remarks that an oar in clear water appears broken, and that apples, seen through a glass are magnified, and Archimedes is said to have published a book "on a Ring seen under water*."

Alhazen also knew that the quantities of refraction differ according to the magnitude of the angle QOM (fig. 27), and moreover that the angles of refraction do not follow the

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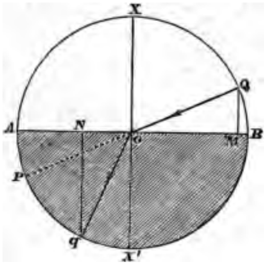
[*] Whewell's *History of Inductive Sciences*, Vol. II. p. 345.

proportion of the angles of incidence, *i.e.* that if, for instance $QOM = 2\,QO'M$, then qON is not $= 2q'O'N'$ (fig. 27).

After reaching this point, there remained nothing to be done with regard to refraction except to go on experimenting and conjecturing till the true law of refraction was discovered, viz. the law in Art. 32 (ii). This was discovered at last by Wildebrord Snell, a Swede, about 1621, but the law was first published by Descartes, who had seen Snell's papers. Snell enunciated the proposition in the following form.

Let Q be the luminous point and QO a ray meeting the surface in O. Join OQ, and describe a circle AQB, centre O, distance OQ. Draw QM perpendicular to the surface,

Fig. 28.



and take $ON : OM$ in the given ratio: draw Nq at right angles to the surface, meeting the circle in q and join Oq, then Oq is the direction of the ray after refraction. The ratio $OM : ON$, *i.e.* the fraction $\dfrac{OM}{ON}$, is usually denoted by the symbol μ, and is called the *Refractive Index* of the media.

If QO be produced to meet the circle in P, the angle POq, or the difference between the angles QOX, qOX'

which are called the angles of incidence and refraction, is called the *Deviation* of the ray QO.

34. This law is called the *Law of Sines;* and the reason of this is easily seen. For if θ and ϕ be the angles QOX, qOX', *i.e.* the angles of incidence and refraction of the ray QO, then $\frac{OM}{ON}=\frac{\sin\theta}{\sin\phi}$; $\therefore$ $\frac{\sin\theta}{\sin\phi}=\mu$, or $\sin\theta=\mu\sin\phi$.

In precisely the same way, if θ' and ϕ' be the same quantity for another ray, $\sin\theta'=\mu\sin\phi'$.

Thus the laws from which all the phenomena of refraction are deduced may be briefly given:

(i) The incident and refracted rays lie in one and the same plane with the normal to point the surface at the point of incidence, and on opposite sides of it.

(ii) If θ and ϕ have the meanings assigned to them, then $\frac{\sin\theta}{\sin\phi}$ or $\frac{OM}{ON}$ is a constant ratio.

Obs. Since on entering a dense medium from a rare one the ray is bent nearer to the normal; $\therefore$ $\frac{OM}{ON}$ or $\frac{\sin\theta}{\sin\phi}$, (*i.e.* μ) is greater than 1, and *vice versâ*. It should be remarked that when a ray of light strikes any refracting substance such as glass or water, &c. some of the light is always reflected, and the larger the angle of incidence is, the more light is reflected.

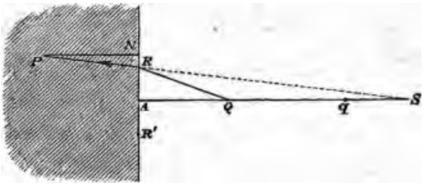
35. Suppose now that a conical shell of rays diverging from Q fall upon the plane refracting surface AB. The rays forming this shell will, by symmetry, all be refracted in just the same manner relatively to the axis QA which is drawn perpendicular to the surface.

Let QR be any one of the rays of our conical shell.

he figure, as before, represents a section by the plane of e paper.)

QR will be refracted in direction RP, nearer to the

Fig. 29.



ormal (by Art. 32). Let RP, produced backwards, meet Q produced in S.

The position assumed by S when QR falls as near as imaginable to QA is called the geometrical focus of the oint Q. Let q be this position.

Take $RP = RQ$ and draw PN perpendicular to the urface.

Then $\frac{AR}{RN} = \mu$ (Art. 33); $\therefore AR = \mu . RN$.........(i);

$$\therefore \frac{AR}{RQ} = \frac{\mu . RN}{RQ} = \frac{\mu . RN}{RP} = \mu . \frac{AR}{RS} \text{ by similar triangles;}$$

$\therefore RS = \mu . RQ$.

If now we suppose our rays to fall on the surface very lose to A, RS becomes $= Aq$,

RQ.......... $= AQ$.

$\therefore \; Aq = \mu . AQ$...........................(ii).

Thus if $\mu = \frac{4}{3}$, $AQ = 3$ feet, then $Aq = \frac{4}{3} . 3 = 4$ feet.

36. By using trigonometry this investigation may be made much shorter.

For if $RQA = \theta$, $RSA = \phi$, $\frac{\sin\theta}{\sin\phi} = \mu$.

But $$\frac{\sin\theta}{\sin\phi}=\frac{RS}{RQ};\quad\therefore\ RS=\mu RQ,$$

or in the limit $$Aq=\mu AQ.$$

If the refraction is from rare into dense μ is > 1 and q is to right of Q, and *vice versâ*.

37. It is necessary to understand clearly what we have proved in Arts. 35, 36. It is this. Suppose that of all the rays which fall from O upon the surface, we select only a *very small* pencil the axis of which falls at right angles to the surface: then all these rays will after refraction proceed as if they were diverging very nearly indeed from one point q on the axis such that $Aq=\mu\,.\,AQ$. This point then, which is called the geometrical focus, would be, for an eye receiving such rays, the image of the point Q. (Compare Reflexion.)

By the principle that light will always return by the same path as that by which it has gone, we see that a pencil *con*verging to q from the left, would, on coming into the rare medium, be bent so as to converge to Q.

Critical angle.

38. It is found that a ray entering a dense medium from a rare medium is always refracted, however great an angle its direction makes with the normal to the surface, but that there is a direction after passing which a ray entering a medium rarer than its own is not refracted, but is entirely reflected. This we can easily see from fig. 29.

Suppose that there the medium below AB is the denser: *i.e.* $\mu > 1$. Then $OM=\mu\,.\,ON$, *i.e.* OM is always $> ON$. But the greatest value of OM is OB; $\therefore$ ON can never be so large as OB or OA, however near OQ comes to OB. Its greatest value in fact $= \frac{1}{\mu} OB$, corresponding to the ray from Q when Q comes to B.

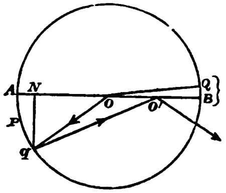
Thus, since light can always retrace its steps, we see that a ray from any point P nearer to A, if $ON = \frac{1}{\mu} OB$, would not be refracted at all, but would be reflected at the surface.

From fig. 30, $qN^2 = Oq^2 - ON^2 = OB^2 - ON^2$.

But $OB = \mu . ON$; $\therefore qN^2 = (\mu^2 - 1)\, ON^2$;

$$\therefore ON = \frac{qN}{\sqrt{\mu^2 - 1}}.$$

Fig. 30.



Therefore, to construct the critical angle for any point q in the dense medium below AB, draw qN perpendicular to the surface, cut off $NO = \frac{qN}{\sqrt{\mu^2 - 1}}$ and join qO; then $NqO =$ critical angle, and Oq is the direction of a ray from Q close to the surface after refraction.

Thus to an eye at q, all objects above the surface would appear to be placed within the cone whose vertical angle $= 2NqO$: while if q be taken as the luminous point all rays such as qO' making a greater incident angle than qO will

not get through to the rare medium at all, but w
reflected internally.

Trigonometrical Explanation.

39. If θ and ϕ be the angles of incidence and
gence, $\sin\theta = \mu \sin\phi$, or $\sin\phi = \frac{1}{\mu}\sin\theta$.

Now if the refraction is from *rare to dense* $\frac{1}{\mu} <$
$\therefore \sin\phi$ is < 1 whatever value θ has, *i.e.* ϕ is possible
values of θ, or a ray can always be refracted into a
medium.

But if the refraction is *from dense to rare*, $\frac{1}{\mu} >$
$\therefore \sin\phi$ is < 1 only so long as $\sin\theta$ is $< \frac{1}{\mu}$ or $\theta <$ s
That is, if the ray falls upon the surface making an
greater than $\sin^{-1}\frac{1}{\mu}$ it will not be refracted. The ra
makes an angle $= \sin^{-1}\frac{1}{\mu}$ comes out along the surfac
refraction.

This angle $\sin^{-1}\frac{1}{\mu}$ corresponds to the angle NqO in f

Thus, from water into air, no ray will be refracted
ing an angle greater than $\sin^{-1}\frac{3}{4}$, or $\sin^{-1}\frac{2}{3}$ if from
into air.

CHAPTER VII.

Miscellaneous.

40. As in reflexion at a plane surface, what was proved true for the single luminous point Q is true of any point of a luminous object. Thus, any point of a luminous object has its Geometrical Focus at a distance from the surface determined by the formula of Arts. 36 and 37.

Let PQX be any luminous object, and BAC the refracting surface denser than that in which is PQX. Then P has its Geometrical Focus at p, where

$$Bp = \mu . BP.$$

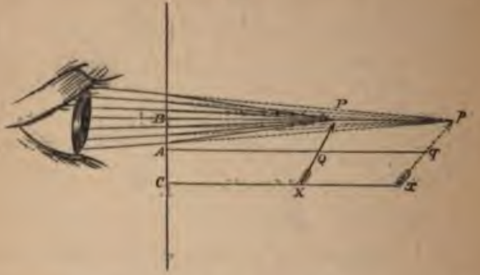
So $$Aq = \mu . AQ,$$

$$Cx = \mu . CX,$$

PB, QA and XC being supposed normal to the surface.

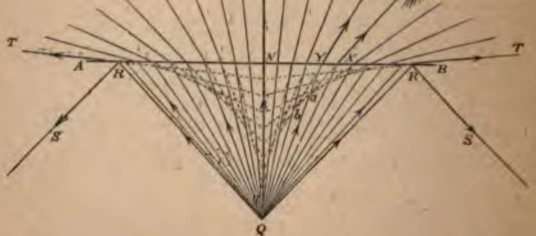
Suppose that PX and px are produced to meet: they will meet in BC produced, since $BP : Bp :: CX : Cx$, and also Bp and Cx are parallel. Now in the corresponding figure for plane reflexion (Art. 9), we were able to take the points p, x, q as the actual images for an eye in *any* position. But here each of these points is to be regarded as the image only when the eye is placed on the axis of that point, so that no eye could possibly see a distinct image pqx of the whole of PQX, unless it were at an infinite distance, when it might be regarded as on the axis of all points of the object equally.

Fig. 31.



The image *really* seen of any point Q by an eye not on the axis, will be seen in exactly the same way by rays all of which come to the eye as if they touched a caustic curve. Figure 32 represents such a caustic supposing the refraction to take place from dense into rare.

Fig. 32.



The rays QR are those which fall upon the surface at the critical angle, and are therefore reflected back again, as

RS, instead of getting through the surface. Thus angle $RQR = 2 \sin^{-1} \frac{1}{\mu}$. A ray falling on the surface the least bit nearer to N than R will be refracted in direction RT, close to the surface. QN the normal ray will of course pass through n the same line, and the other rays will all be refracted as n the figure. Now it can be proved, as we said in the corresponding part of reflexion, that if we produce the directions of these rays after refraction backwards, they will cut the axis QN in points the consecutive distances of which continually increase.

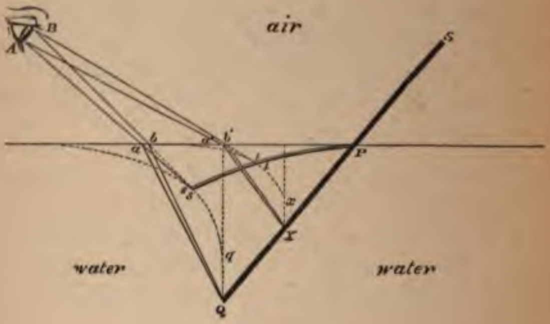
Thus there will be a *virtual* caustic curve formed instead of a *real* one. If E then be an eye in any position, and QXx, QYy be the extreme rays of the very small pencil which come to it, all the rays of this pencil, viz. those falling on the surface between X and Y, will come to it as if from points on the caustic curve between a and b, and so a blurry mage of Q will be seen by the eye E at ab. This caustic urve, then, has its cusp, as before, in q the geometrical ocus, and touches the surface at R and R.

The reader is recommended to draw for himself the austic formed by refraction from a rare medium into a dense medium.

Figure 33, then, will shew why a stick partly in and artly out of the water appears bent where it enters the water.

Let Q and X be points of the stick, q and x their Geometrical foci. Draw the caustic curves of these points Q nd X. Then an eye at E will see a blurry image of Q at s, and a blurry image of X at *tt* where the rays to the eye from Q and X touch the virtual caustics of those points. The same will be true for every point of the stick. Thus the eye will see the stick as in the figure: the image appearing curved. This effect may be noticed when standing above water of an uniform depth. The bottom appears to

Fig. 33.



slope gradually up to the water as it recedes from the place where we are standing.

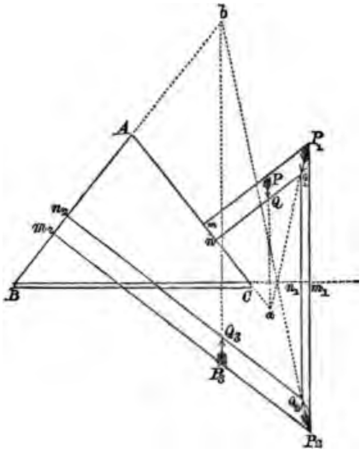
41. We will now give an example which combines the principles of plane reflexion and plane refraction. We shall assume that the image in each refraction is at the Geometrical Focus.

Example. A pencil of light passes through a triangular prism of glass, after having suffered one internal reflexion at one face of the glass. To trace its course from the origin.

Let *ABC* be a section of the prism made by the plane of the paper. Let *BC* be silvered, so as to reflect light. Let *PQ* be the object. Then

i) By *refraction* at AC, PQ will have an image P_1Q_1 of itself on the normals to AC, such that $mP_1=\mu\,.\,mP$, $nQ_1=\mu\,.\,nQ$. By Art. 44 it will be so inclined as that it meets PQ produced in a on the surface AC.

Fig. 34.



(ii) After entering the glass the light goes as if it came from P_1Q_1 until it falls upon the *reflecting* surface BC, when, by the principles of plane reflexion, it will have its image at P_2Q_2, on the normals to BC, where $m_1P_2=m_1P$ and $n_1Q_2=n_1Q_1$.

(iii) After striking BC the light goes as if it came from P_2Q_2 until it emerges at AB, where by refraction it has an image P_3Q_3 such that

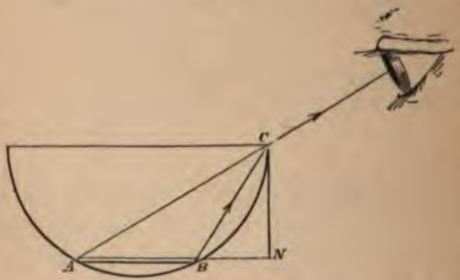
$$M_2P_3=\frac{1}{\mu}\,.\,m_2P_2 \text{ and } n_2Q_3=\frac{1}{\mu}n_2Q_2.$$

As before, P_3Q_3 and P_2Q_2 would meet in BA produced.

It would be a very good exercise for the reader if he would trace the course of a small pencil of rays from Q to an eye placed anywhere in the plane of the paper. He must remember that the Geometrical Focus is taken for the image at the refractions, and that at the refractions he must therefore only consider rays falling nearly at right angles.

42. *A coin is placed at the bottom of an empty hemispherical basin of given radius, and is just not visible to an eye looking over the edge. When the basin is filled with water, the whole of the coin is just visible to an eye in the same position. Find the diameter of the coin.*

Fig. 35.



Let AB be the diameter of the coin $= 2x$. Since at first it is just not visible to the eye, therefore the eye lies in AC produced. Since, when the basin is full it is just visible entirely, therefore the ray from B through C is just refracted to the eye.

Thus $\theta =$ angle of incidence of $BC = BCN$,

$\phi =$ angle of emergence of $BC = ACN$,

$$\therefore \frac{BN}{BC} = \mu \frac{AN}{AC},$$

where μ = index of refraction from water into air;

$$\text{i.e.}\ \frac{a-x}{\sqrt{(a-x)^2 + a^2 - x^2}} = \mu \cdot \frac{a+x}{\sqrt{(a+x)^2 + a^2 - x^2}},$$

or
$$\frac{a-x}{\sqrt{2a(a-x)}} = \mu \cdot \frac{a+x}{\sqrt{2a(a+x)}};$$

whence
$$x = a \cdot \frac{1-\mu^2}{1+\mu^2}.$$

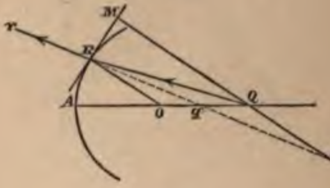
This problem was explained as early as A.D. 140 by Ptolemy, the Alexandrian astronomer.

CHAPTER VIII.

Refraction at a spherical surface.

43. To find the Geometrical Focus of a pencil of rays after direct refraction at a concave spherical surface.

Fig. 36.



Let Q be the luminous point, and let O be the centre. Then QOA is the axis of the pencil from Q which falls directly on the surface. Of the pencil consider any ray QR which falls on the surface at R; let Rr be its direction after refraction, nearer to the normal OR than before. Let RM be the plane touching the spherical surface at R, (the whole figure being, as usual, a section by the plane of the paper): let QM be a perpendicular to this plane. Let MQ and rR produced backwards meet in S.

Since RM is a tangent plane at R, the refraction which takes place there is the same as would take place at the plane refracting surface *RM*.

$$\therefore\ MS = \mu \,.\, MQ \text{ by (Art. 39. i.)}$$

But, by similar triangles, ROq, SQq,

$$\frac{QS}{Qq} = \frac{OR}{Oq},$$

or
$$\frac{MS - MQ}{Qq} = \frac{OR}{AQ - AO}.$$

Substituting for MS

$$\frac{MQ(\mu - 1)}{AQ - Aq} = \frac{AO}{Aq - AO}.$$

In the limit, when R moves up to A, MQ becomes AQ,

$$\therefore\ \frac{(\mu - 1) \,.\, AQ}{AQ - Aq} = \frac{AO}{Aq - AO},$$

whence
$$\frac{\mu}{Aq} - \frac{1}{AQ} = \frac{\mu - 1}{AO} \ldots\ldots\ldots\ldots\ldots(1).$$

Thus, if a number of rays from Q fall *on the surface very nearly* at A, they will, after refraction, proceed as if they came almost exactly from q, where the position of q is found from the formula (i).

Example. $\mu = \frac{3}{2}$, $AQ = 6$ feet, $AO = 4$ feet,

$$\frac{3}{2}\,\frac{1}{Aq} = \frac{1}{8} + \frac{1}{6} = \frac{7}{24},$$

$$\therefore\ AQ = \frac{36}{7} = 5\tfrac{1}{7} \text{ feet.}$$

If the surface were convex the formula would be found to be

$$\frac{\mu}{Aq} - \frac{1}{AQ} = -\frac{\mu - 1}{AO}.$$

44. If now we suppose Q to be very distant, a star, f instance, i.e. if we suppose a pencil of *parallel* rays to f on the surface close to A and parallel to the diamet through A, we may in our formula neglect $\frac{1}{AQ}$, and have then

$$\frac{\mu}{AF}=\frac{\mu-1}{AO} \text{ or } AF=\frac{\mu}{\mu-1}AO,$$

where F is the position which the point q assumes in su a case. As in reflexion, F is called the "*Principal Focu* Thus for parallel rays from air into water,

$$AF=\frac{\frac{4}{3}}{\frac{4}{3}-1}AO=4\,.\,AO.$$

On the whole then we shall have, as in the correspor ing case of reflexion, the following rules for determini the position of our Geometrical Focus.

(i) Q and q must lie on the same normal, since all t rays from Q are supposed to proceed as if frc q, after refraction, and since the direction of t normal ray is not altered at all.

i.e. The Geometrical Focus of any point must lie on t normal to the surface from that point.

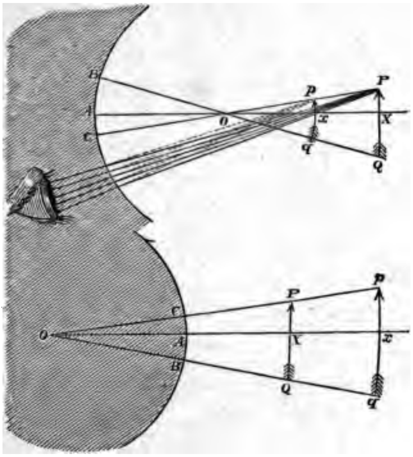
(ii)
$$\frac{\mu}{Aq}-\frac{1}{AQ}=\pm\frac{\mu-1}{AO}$$
according as the surface is concave or convex.

(iii) Q and q move in the same direction, since as A increases, so must Aq from the formula.

(iv) Hence Q and q always lie on opposite sides of tl principal focus.

45. Let *PQX* be an object in front of the spherical refracting surface *BAC*.

Fig. 37.



By the last article the Geometrical Focus of *P* is at *p*, fig. 38 (i), where

$$\frac{\mu}{Cp} - \frac{1}{CP} = \frac{\mu - 1}{CO},$$

that of *Q* is at *q* where

$$\frac{\mu}{Bq} - \frac{1}{BQ} = \frac{\mu - 1}{BO},$$

that of *X* is at *x* where

$$\frac{\mu}{Ax} - \frac{1}{AX} = \frac{\mu - 1}{AO}.$$

In fig. (ii).

$$\frac{\mu}{Cp} - \frac{1}{CP} = -\frac{\mu - 1}{CO},$$

$$\frac{\mu}{Bq} - \frac{1}{BQ} = -\frac{\mu - 1}{BO},$$

$$\frac{\mu}{Ax} - \frac{1}{AX} = -\frac{\mu - 1}{AO}.$$

46. There is one more point to notice. In t mulæ for concave and convex refracting substanc observe that the only difference is in the sign of th hand side of the equation. The two formulæ will co if we make this assumption about the *signs* of our lin lines measured from the origin *A opposite* to the di in which the light is travelling are to be regarded as and *vice versâ*.

Applying this to figure 36, we see that all the line drawn from A are opposite to the direction of the i light, and are all therefore positive. This case then regarded as the standard case.

But if the reader will draw the corresponding fig a convex substance he will see, as in fig. 37 (ii), that drawn from A in the negative direction. This accou the difference in sign in the two formulæ.

47. It is sometimes more convenient to have a f referred to the centre as origin, instead of one refe A.

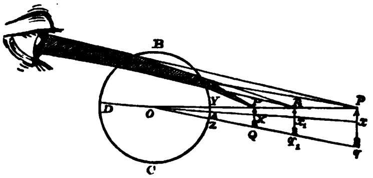
From figure 36, $Aq = AO + Oq$, $AQ = AO +$

Substitute in the formula $\frac{\mu}{Aq} - \frac{1}{AQ} = \frac{\mu - 1}{AO}$, and

we obtain this $\frac{1}{Oq} - \frac{\mu}{OQ} = \pm\frac{\mu - 1}{AO}$, for concave or surface. Fig. 36 is clearly the standard case for the origin O, since then all the lines OA, OQ, Oq are po

48. We will, as a good example, trace the effects of refraction through a sphere of some substance denser than the surrounding medium.

Fig. 38.



We will suppose that the rays after each refraction come as if accurately from the Geometrical Focus.

(i) Let a small pencil of rays from the point P of the object PQ fall upon the surface BAC and be refracted into the sphere. After entering the sphere these rays will proceed as if they came from p_1 on the normal Op_1 where

$$\frac{1}{Op_1} - \frac{\mu}{OP} = -\frac{\mu - 1}{AO} \text{.........} (A).$$

(ii) These rays, which come as if from p_1, will, on being refracted out into the original medium, proceed as if they came from p, where, since the refraction is from dense to rare,

$$\frac{1}{Op} - \frac{\frac{1}{\mu}}{Op_1} = \frac{\frac{1}{\mu} - 1}{AO},$$

$$\text{or } \frac{1}{Op_1} - \frac{\mu}{Op} = \frac{\mu - 1}{AO} \text{.........} (B).$$

To trace the course of the rays through the sphere to the eye, we do just as in reflexion,

(1) Join the last image to the eye by straight lines and shade from the eye to the last surface.

(2) From the points where these lines cut the last surface, to the last image but one, and shade to the last surface but one.

(3) From these last points to the object and shade.

The shaded part represents the rays from the point P to the eye.

Substituting [illegible] from [illegible]

$$[illegible]$$

or

$$[illegible]$$

[illegible]. The three following examples are recommended to the reader's attention:

(1) The last example, supposing that instead of a sphere, the refracting solid is a hemisphere, and that the light is incident upon the curved surface.

(2) A glass sphere is divided into two hemispheres, the index of refraction in one being different from that in the other. Find the positions of the images as in Art. [illegible].

(3) An eye is placed close to the surface of a sphere of glass ($\mu = \frac{3}{2}$) which is silvered at the back. Shew that the image which the eye sees of itself is $\frac{3}{5}$ natural size.

CHAPTER IX.

Refracting Plates and Refracting Prisms.

A Refracting Plate is a portion of a refracting n bounded by two parallel plane surfaces, as a pane s.

ppose a ray of light enters and emerges from a refract- ate. Then if we recollect the principle that light exactly retrace its steps and that the sides of the re parallel, we see that its direction on emergence e exactly parallel to its direction before entering the If a ray passing through any number of plates in contact come to a plate composed of its original m, its course in this plate will be parallel to its l course.

means of this experimental fact and the principle l it can be proved that if

index of refraction from medium *A* into medium *B*,

....*A*................*C*,

n $\frac{\mu_2}{\mu_1}=$ index from *B* into *C*.

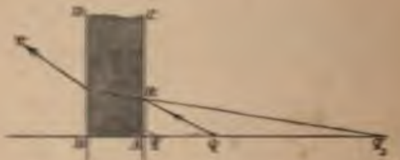
collecting that $\mu=\frac{\sin\theta}{\sin\phi}$, the reader can easily prove nself that if $\mu_1\ \mu_2\ \mu_3\ldots\ldots\mu_n$ be the indices of refraction from A into B, B into C, C into D...... and if μ = index from A into the last, then

$$\mu = \mu_1 \, \mu_2 \, \mu_3 \cdots\cdots \mu_r.$$

51. To find the geometrical focus of a pencil of rays refracted through a plate (index $= \mu$).

Let AC and BD be the sides of the plate: let $QRST$ be the course of a ray. After refraction *into the glass* at the first surface at R we have the rays diverging from a focus q_1,

Fig. 39.



where

$$Aq_1 = \mu \, . \, AQ \ldots\ldots \text{(i)}$$

Art. 36 (ii). The rays now form a diverging pencil in the glass, from q_1: therefore after refraction *out of the glass* at S into the air, they diverge as if from a focus q, where

$$Bq = \frac{1}{\mu} Bq_1 \, ;$$

$$\therefore Bq_1 = \mu \, . \, Bq \ldots\ldots\ldots\ldots \text{(ii)}.$$

Subtracting (i) from (ii)

$$AB = \mu \, . \, AB + \mu \, . \, Aq - \mu \, . \, AQ,$$

$$\therefore Aq = AQ - \frac{\mu - 1}{\mu} AB.$$

Here we have assumed as usual, what is only approximately true, that the rays come accurately from the Geometrical Focus.

It can be shewn that when a ray is refracted out ıe medium into another, as the angle of incidence ıses, so also does the *Deviation.* Art. 34. This prin- which admits of an easy Geometrical proof, will be of use in our investigations into the properties of prisms.

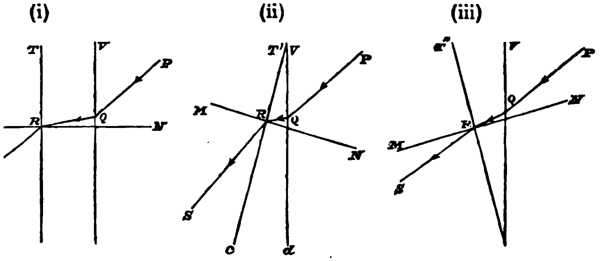
prism is a portion of any refracting medium, bounded o plane surfaces inclined at any angle to one another.

he inclination of these faces to one another is called *Refractive Angle.*"

he line in which these faces meet is called the "*edge*" prism.

e will now prove that "*A ray passing through a prism nedium denser than the surrounding medium is always l from the edge of the prism.*"

Fig. 40.



Let $PQRS$ (fig. 40, (i)), be a ray passing through a plate bounded by parallel sides TR, VQ, and let RS be its direction after emergence.

) By turning RT round into the position RT', we make our plate into a prism whose edge is turned towards $T'V$. The angle which QR makes with the normal has now increased, therefore the deviation has increased, *i.e.* the angle which *RS* makes with the normal has increased still more, and the ray therefore emerges as *RS'* nearer to the thick end of the prism than *PQ*. If the deviation did not increase it would emerge just parallel to *PQ*, *i.e.* it would not be bent at all.

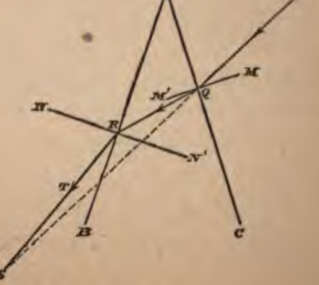
(iii) Next we can turn our plate into a prism with the thick end towards *T* and *N*. The angle which *QR* makes with the normal has now decreased, therefore the angle which the emergent ray makes with the normal has now decreased still more, therefore the ray comes out as *RS''* nearer again to the thick end.

Thus in every case the emergent ray is turned from the edge.

53. The result of the last Article enables us to prove that "A ray passing through a prism is always turned to the edge of the prism, when the medium of which the prism is composed is denser than the surrounding medium."

54. We will now find the value of the deviation of a ray after refraction through a prism.

Fig. 41 (i).



(i) Let $PQRS$ be the course of a ray: let the original and new directions of the ray meet in S.

Then $\quad Deviation = R\hat{S}Q = R\hat{Q}P - S\hat{R}Q = R\hat{Q}A + 90^{\circ}$
$- PQM - (90 - \hat{S}RN + A + \hat{R}QA)$

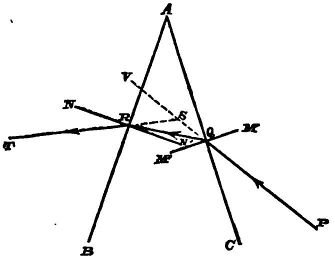
$$= T\hat{R}N - P\hat{Q}M - A$$

= angle of emergence − angle of incidence − refractive angle of prism for rays above the normal.

(ii) $Deviation = TSV = 180 - RSQ = 180 - T\hat{R}Q + S\hat{Q}R$
$= 180 - 90 + TRN - A - RQA + (RQA - 90 + PQM)$

$$= TRN + PQM - A$$

Fig. 41 (ii).



= angle of incidence + angle of emergence − angle of prism for rays below the normal.

55. Now it can be proved that when a ray makes the angle of emergence equal to the angle of incidence, the deviation is less than for any other ray.

If then D = deviation, ϕ and ψ be the angles of incidence and emergence, and A the refractive angle of the prism,

$D = \phi + \psi - A$. When therefore the ray has minimum deviation, $D = 2\phi - A$.

And if ϕ' and ψ' be the angles of refraction at the first, and incidence on the second surface, $\phi' + \psi' = A$, or here $2\phi' = A$:

$$\therefore \mu = \frac{\sin \phi}{\sin \phi'} = \frac{\sin \dfrac{D + A}{2}}{\sin \dfrac{A}{2}} \text{ ....(1).}$$

From this we can determine the refractive index of any substance which can be made into a prism. Measure the minimum deviation D, and the refractive angle A and substitute in (1).

The fact stated at the beginning of this Article can be proved either by the aid of Trigonometry (Parkinson's *Optics*, Art. 94), or Geometrically (*Cambridge Messenger of Mathematics*, Nov. 18[illegible]).

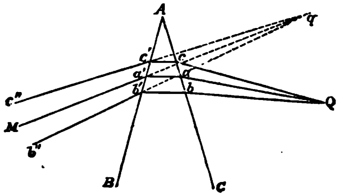
56. The rays close to the ray of minimum deviation on either side have smaller deviation than those more remote. That is, on emerging at the other side of the prism, such rays would form a more compact beam than those elsewhere, that is, an eye placed so as to catch such beams would see stronger light than an eye elsewhere.

This is of the very greatest importance in the discussion for instance on the causes of the Rainbow.

We can deduce a convenient method of determining the position of minimum deviation, i.e. the position in which the prism must be placed so that our eye may receive the rays close to that of minimum deviation, practically. Prick a small hole in a sheet of paper, and observe the spot of through the prism. Then turn the prism about in the plane perpendicular to its edge: as you turn it one way you will see the image move upwards; as you turn it the other, it will move down. But in one certain position it will move down whichever way the prism is turned. This position then is the position of minimum deviation: since the meaning of the experiment is, that on passing this position either way the deviation increases.

57. We have said that the rays which form the very small beam close to the ray which has minimum deviation emerge so that their deviations have been very nearly the same. Since we are considering only a very small beam

Fig. 42.



we may suppose that $b'c' = bc$. If then the deviations of the three rays in the figure were exactly the same, the emergent rays would be just as much inclined to one another as the incident rays were, and therefore, since $b'c' = bc$ practically, they would meet at q where $a'q = aQ$.

Now the deviations are so very nearly the same, and $b'c'$ is so very nearly $= bc$, if we let the rays fall close to the edge of the prism, that the emergent rays of this small beam do very nearly indeed emerge as if from a single point.

If we take a prism with a very small angle A, this will, of course, be still more nearly true.

58. We know that deviation $= \phi + \psi - A$

where
$$\sin\phi = \mu \sin\phi',$$
$$\sin\psi = \mu \sin\psi'.$$

Suppose the rays incident on a prism with a smal fractive angle very nearly perpendicular to the first surfa

Then
$$\left.\begin{array}{l}\phi = \mu\phi \\ \psi = \mu\psi\end{array}\right\} \text{very nearly;}$$

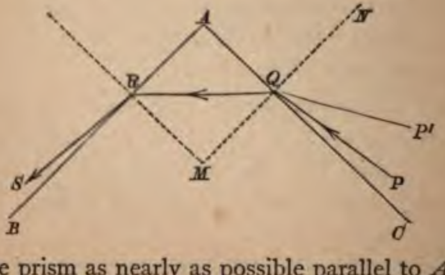
since ϕ' and ψ' are so very small, and therefore
$$\left.\begin{array}{l}\sin\phi' = \phi' \\ \sin\psi' = \psi'\end{array}\right\} \text{very nearly;}$$
$$\therefore \phi + \psi = \mu(\phi' + \psi') = \mu A;$$

$\therefore$ deviation $= (\mu - 1) A$ very nearly $=$ constant quantity, the deviations for the rays making the small beam are same, and therefore the same result follows as in last Art

59. Suppose that we wish to construct a prism that no ray shall be able to get through it.

Let ABC be a prism such that a ray from P, w

Fig. 43.



enters the prism as nearly as possible parallel to AC, co out as RS, as nearly as possible parallel to AB.

Then, clearly, each of the angles MQR, MRQ is the critical angle of the medium.

But $M\hat{Q}R + M\hat{R}Q = A$, i. e. 2 . critical angle $= A$.

Now suppose a ray came from P', so that angle $P'QN$ is $< PQN$. Then angle MQR is decreased, and therefore QRM is increased, i. e. the ray QR makes an angle with the normal AB greater than the critical angle, and will therefore not be refracted out at R, but will be internally reflected.

Therefore the prism through which no ray can pass is that where the refractive angle of the prism = twice the critical angle of the medium.

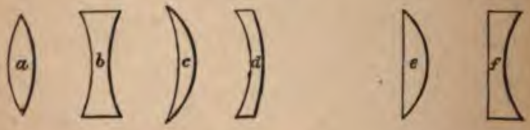
CHAPTER X.

Lenses.

60. A lens is a portion of a refracting medium bou ed by two spherical surfaces: the straight line joining t centres being called the *axis* of the lens. This defini includes those lenses in which one surface is plane, sin plane may be regarded as a sphere of an infinite radius.

Lenses are distinguished by different names, accor to the nature of their surfaces. In the following figures light is supposed to pass from right to left. Thus there six kinds of lenses; or, including a prism and a plate u the definition, there are eight.

Fig. 44.



Thus (*a*) is called a *double convex* lens.
(*b*) is called a *double concave* lens,
(*c*) and (*d*) are called *convexo-concave* le
(*e*) is called a *convexo-plane* lens,
(*f*) is called a *concavo-plane* lens.

61. One of the earliest notices of a lens is in the writings of Seneca, A.D. 64 (*Nat. Quest.* lib. I. ch. 7), in which he states the fact that objects seen through a ball of glass full of water appear larger. This has already been explained (Art. 49). Pliny in A.D. 79, speaking of the manufacture of glass at Sidon, says, "siquidem etiam *specula* excogitaverat." This word "*specula*" probably refers to lenses, as Pliny states that the art of grinding glass was then known.

After an interval of some centuries we arrive at the time of Alhazen, the Arabian mathematician, A.D. 1100. He has attempted an anatomical description of the eye, and has assigned the respective functions of the several parts of vision. To do this as he has he must, as will be seen, have had a good general idea of the effect of lenses.

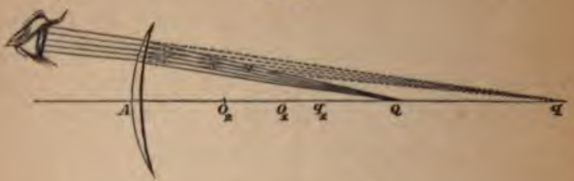
Jean Baptiste Porta of Naples, A.D. 1445, states that if a small orifice be made in the shutter of a room from which all other light is excluded, and if a small convex lens be placed at the aperture, external objects will appear on a sheet of paper, placed at the proper distance, so distinct as to be instantaneously recognized. We will now explain this latter fact. It must be noticed that although the facts were noticed, and could in a rough way be explained, yet until the discovery of the law of refraction by Snell in 1621, a real explanation was wanting.

62. A pencil is said to be *directly* refracted through a lens when the axis of the pencil is coincident with the axis of the lens.

The process of finding the geometrical focus of a pencil after direct refraction through a lens is precisely similar to that used for a plate (Art. 52).

Let Q be the point of light from which a pencil falls directly on the lens. The axis of the pencil is therefore QAB, a common diameter of the spherical surfaces.

Fig. 45.



Let (r) and (s) be the radii of the spherical surfaces (A) and (B). Let q_1 and q be the geometrical foci of the pencil after first and second refractions, the first being *into* glass, the second *out* of it.

Let O_1 and O_2 be the centres of the surfaces (A) and (B). Then q_1 and q will have positions as in the figure.

From the *first* refraction we obtain

$$\frac{\mu}{Aq_1} - \frac{1}{AQ} = \frac{\mu - 1}{r},$$

from the *second*,

$$\frac{\frac{1}{\mu}}{Bq} - \frac{1}{Bq_1} = \frac{\frac{1}{\mu} - 1}{s} \text{ or } \frac{1}{Bq} - \frac{\mu}{Bq_1} = -\frac{\mu - 1}{s}.$$

If the thickness of the lens is so small that Aq_1 may be regarded as equal Bq_1 and $Aq = Bq$, then by adding these equations

$$\frac{1}{Aq} - \frac{1}{AQ} = (\mu - 1)\left(\frac{1}{r} - \frac{1}{s}\right),$$

which determines the position of q. As before, Q and q are called "*conjugate foci*" with respect to the lens.

Suppose AQ to be very large, i.e. suppose the rays to ll parallel, and let q then come to F. Then

$$\frac{1}{AF}=(\mu-1)\left(\frac{1}{r}-\frac{1}{s}\right).$$

The point F is *then* called the "*principal focus*" of the lens nd AE its "*focal length*." This is usually written f. Thus

$$\frac{1}{f}=(\mu-1)\left(\frac{1}{r}-\frac{1}{s}\right); \quad \therefore \frac{1}{Aq}-\frac{1}{AQ}=\frac{1}{f},$$

where q is the geometrical focus for any point Q.

If r and s are equal

$$\frac{1}{f}=\pm\frac{2(\mu-1)}{r}$$

or *double concave or convex.*

63. It should be noticed that we have found this quation from the case of a *concavo-convex* lens, that being aken as the standard case because the quantities r and there are both positive. If we took the case of a *double nvex* lens the formula would be

$$\frac{1}{Aq}-\frac{1}{AQ}=(\mu-1)\left(-\frac{1}{r}-\frac{1}{s}\right),$$

und from the first by putting $-r$ for $+r$ by the principles Art. 22.

For a concavo-plane lens since s is then $=\infty$

$$\frac{1}{Aq}-\frac{1}{AQ}=(\mu-1)\frac{1}{r}.$$

For a convexo-plane lens since s is then $=\infty$

$$\frac{1}{Aq}-\frac{1}{AQ}=-(\mu-1)\frac{1}{r}.$$

For a double concave lens

$$\frac{1}{Aq} - \frac{1}{AQ} = (\mu - 1)\left(\frac{1}{r} + \frac{1}{s}\right).$$

Obs. In any given lens the position of the focus is the same whichever face be turned towards the incident light. For, neglecting the thickness of the lens, our formula is

$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{r} - \frac{1}{s}\right).$$

Now if we turn our lens round, the formula must be written

$$\frac{1}{f'} = (\mu - 1)\left(\frac{1}{(-s)} - \frac{1}{(-r)}\right) = \frac{1}{f}.$$

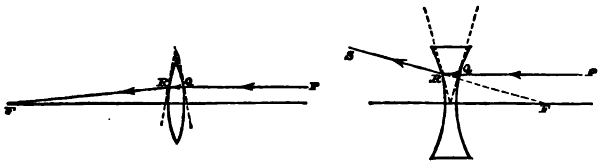
64. If we were told the nature of our lens, i.e. the radii of the surfaces which formed it, we could from

$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{r} - \frac{1}{s}\right)$$

determine on which side of the lens the principal focus lay, by noticing the *sign* of the right-hand quantity. But a better rule can be given thus. If the lens be thickest in the middle, the principal focus is on the side away from the incident light, i.e. focal length is negative. If the lens be thinnest in the middle, the reverse is the case. It can be proved thus.

Let tangent planes be drawn at Q and R: then the course of the ray is the same as it would be through the prism formed by these planes: and therefore, since the thicker part of the lens is in the same direction as the thicker part of the prism, and since a ray is bent towards the thicker part of the prism (Art. 53), therefore the ray is bent towards the thicker part of the lens. Therefore the position of the principal focus must be as given in the rule.

Fig. 46 (i). (ii)



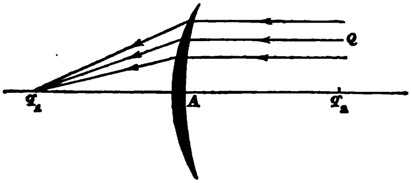
The reader may feel some difficulty in reconciling this with fig. 45. The next Article will, however, explain it.

65. We will investigate this equation $\frac{1}{Aq} - \frac{1}{AQ} = \frac{1}{f}$ in just the same manner as we did in the case of spherical reflexion.

(1) Since $f =$ constant quantity, as AQ increases or decreases, so must Aq, i.e. Q and q always move in the same direction.

Let q_1 be the principal focus of the lens for rays coming from the right. Take $Aq_2 = Aq_1$; then q_2 is the principal focus for rays from the left (*Obs.* Art. 63).

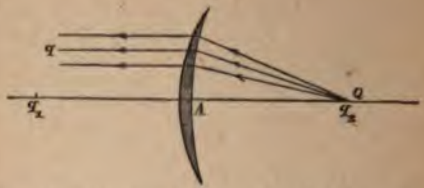
Fig. 47 (i).



As Q moves, from an infinite distance to the right, towards A, q_1, which was at q, moves off to the left (Art. 64).

(2) When Q comes to q_2, the image q has moved off t the left to an infinite distance, since q_2 is the princ pal focus for rays from the left. (ii)

(ii)

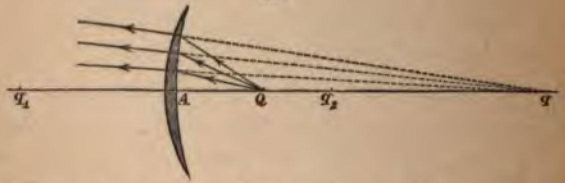


(3) As Q still approaches (A) the image q as it wer turns the corner at infinity, and appears again t the right of q_2. (iii)

(q is now only a *virtual* image.)

This is the case taken for fig. 45.

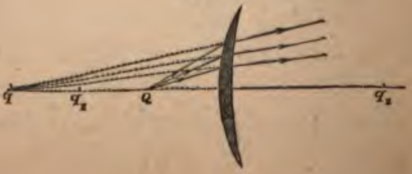
(iii)



(4) Q and q approach A together, and arrive there s multaneously: for, when $AQ = 0$, so does Aq.

(5) As Q goes off to the left, so does q. (iv)

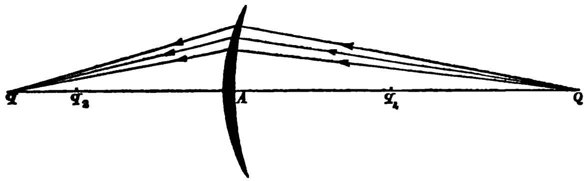
(iv)



(6) And, lastly, when Q arrives at q_1, q has again gone away to an infinite distance.

Figure (v) shews the usual case where Q is a considerable distance from A; intermediate between figs. (i) and (ii).

(v)



This then is the explanation which we promised in Art. 61 of Porta's experiment. The reader is strongly advised to study this Article thoroughly, and to work out the same investigation for a double concave lens.

66. If the focal length of a lens is small, the parallel rays from the object are clearly more bent by the lens than when the focal length is large. That is, since f is the focal length, $\frac{1}{f}$ is a measure of the amount that the rays are bent by the lens. It is therefore called the "*Power*" of a lens.

Suppose now that we had three lenses C_1, C_2, C_3 with a common axis. Let Q be a luminous point on the axis, and let q_1, q_2 and q be the geometrical foci of the refracted pencil at these lenses respectively.

Then from the *first* refraction $\frac{1}{C_1q_1} - \frac{1}{C_1Q} = \frac{1}{f_1}$.

Then from the *second* refraction $\frac{1}{C_2q_2} - \frac{1}{C_1q_1 + a_1} = \frac{1}{f_2}$.

Then from the *third* refraction $\frac{1}{C_3q} - \frac{1}{C_2q_2 + a_2} =$

where a_1 a_2 are the distances between C_1 and C_2, a tween C_2 and C_3, and f_1, f_2 and f_3 are the focal len the lenses.

If we eliminated C_1q_1 and C_2q_2 between these eq we could find C_3q. For instance, if $a_1 = a_2 = 0$, i.e. lenses are in contact and very thin, we have by additi

$$\frac{1}{C_3q} - \frac{1}{C_1Q} = \frac{1}{f_1} + \frac{1}{f_2} + \frac{1}{f_3}.$$

Thus the power of the three lenses = sum of the po the three lenses separately. Or, instead of the three we might have one lens such that if F were its focal l then $\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2} + \frac{1}{f_3}$, and the effect would be the Similarly for any number of lenses, except that then, for rough calculations, we could not neglect the thickn

67. The focal length of a lens can be found e mentally thus.

The rays from the sun, if allowed to fall directly lens, would, since they may be called parallel, form refraction a convergent conical beam to the principal They do not all pass accurately through one point, the beam be intercepted by a screen parallel to the pla the lens, the little circle of light formed on the scree be smallest at some one position of the screen. This circle is to be taken as the principal focus. Measur distance from the lens to the screen, and the focal len then found.

There is one other remark we wish to make o appearance of an object as seen through a double co

s. If, after reading the reasoning and looking at the res which shew that an image is formed nearer to the than an object really is, the student takes a double cave lens in his hand, he will be surprised to see that image is apparently a *very long way off.* The following planation may perhaps suffice.

(1) He knows the real size of the object before looking at it through the lens;

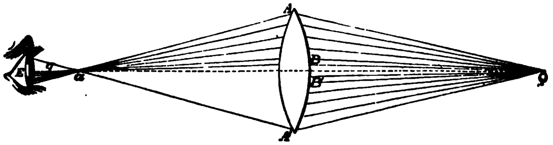
(2) He knows that if an object of a given size is far off it looks smaller than when near;

(3) He sees that the image is small compared to the known size of the object;

therefore his mind at once says that the image is formed a long way off. On looking through the lens for some time, however, this delusion entirely disappears.

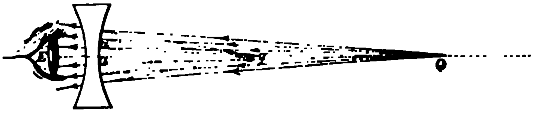
On looking at any object through a double concave and a double convex lens successively, one will notice that the image is far more distinct in the double concave lens than in the double convex: and the reason of this is easily seen if we recollect what has been our fundamental supposition in every one of our investigations; viz. that in asserting that an image is formed at a point which we call the "Geometrical focus," we consider those rays, and those rays *only*, which fall on the surface or lens close to the axis. Rays that come to the eye from the outer parts of the lens tend to make the image indistinct.

Fig. 48.



Thus, in the double convex lens, an eye E would, since it catches all the rays that come from the lens, see an image not quite distinct: for the rays from A and A' cut the axis at a, whereas from B and B' they cut in q. It would see the best image at q, since, as we have said when treating on caustics, the rays are thickest about there.

Fig. 49.

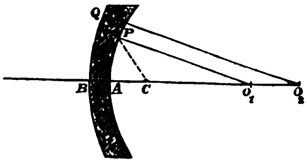


But in the double concave lens, since the rays *spread* after refraction, an eye of the same size as before, even if put close to the lens, will catch only those rays which fall very close to the axis, between a and a', and will therefore see a very distinct image at q.

68. Centre of a lens.

If we join the points of two circles where the radii of the circles are parallel, the joining line when produced will always pass through a fixed point on the line joining their centres.

Fig. 50.



Thus if P and Q be *any* two points on the spherical surfaces such that the radii O_1P, O_2Q are parallel, QP will ss through some one point C on the axis. This point is called the "*centre*" of the lens.

Now

$$\frac{CO_1}{CO_2}=\frac{O_1P}{O_2Q} \text{ i.e. } \frac{AO_1-AC}{BO_2-BA-AC}=\frac{AO_1}{BO_2},$$

ıence $$AC=\frac{AO_1 . BA}{BO_2-AO_1}.$$

'e see from this value of AC that the position of C always the same for the same lens, but depends upon the ıape of the lens.

(1) If the curvatures are in the same direction, i.e. if the centres are both to the right or both to the left, AC is positive or negative and bigger than BA, unless BO_2 is equal to or greater than $2AO$,

i.e. C lies outside the lens.

(2) If the curvatures are in opposite directions, AC is negative and always less than BA. That is, C always lies inside the lens, and if the thickness BA is very small, C may be taken in the middle point.

(3) If one surface be plane, $AC = O$ or $-BA$. Therefore C lies on the curved surface.

69. It is very necessary to gain a precise idea of what ıis centre is. Suppose that a ray of light met the first surıce at P in such a way that it was refracted in direction $^{\text{P}}Q$. Since at Q the tangent plane is parallel to that at P, he effect on the whole would clearly be the same as if the ay had been refracted through a plate, i.e. it would come ›ut of the lens parallel to its former direction.

We shall, for the future, consider only such lenses as hose in (2), and suppose that they are very thin. Then, by vhat has just been said, any ray that passes through the ıiddle point of the lens is refracted without any deviation.

70. We have hitherto considered lenses only in ca when the luminous point was situate on the axis. But, course, when we look at any object through a lens th can be but one point on the axis. It is therefore v necessary to know where the images of the other points the objects are formed. We are able to do this by the u of the centre. As however the investigation involves co siderable analysis, and is not necessary to our future wo we will only give the result, which is extremely simple. is, as usual, only *very approximately* true.

Fig. 51 (i).

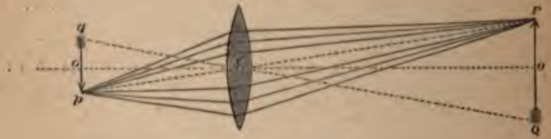
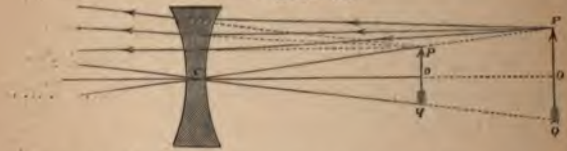


Fig. 51 (ii).



Let QCq be the ray, from Q the luminous point no the axis, which passes through the centre of the lens.

Then the result is this.

The image of Q may be regarded as formed at q, w the same formula obtains as for direct refraction, viz.

$$\frac{1}{Cq} - \frac{1}{CQ} = \frac{1}{f}.$$

As usual we consider only a very small pencil c about the axis QCq.

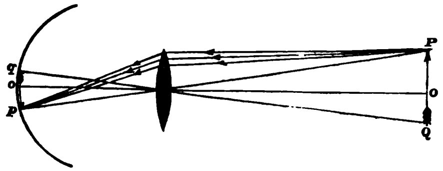
The image therefore will appear inverted in fig. 51

and upright in fig. 51 (ii). In both cases it will appear slightly curved, for from the equation $\frac{1}{Cq} - \frac{1}{CQ} = \frac{1}{f}$ we obtain $\frac{Cq}{CQ} = \frac{1}{f + CQ}$, which shews that although Cq increases with CQ, yet the proportion of Cq to CQ decreases. If the incident rays were parallel we have $Cq = f$, and therefore the image is part of a circle whose radius $= f$.

71. We are now in a condition to see how the use of a lens can assist vision; and without at present giving an accurate description of the human eye, we will merely say that it is in effect one double convex lens which forms images, and a retina or network of nerves which receives the images and transmits the impressions which they convey to the brain. The retina is spherical.

If now the object POQ be just at the right distance for most distinct vision, i. e. so that the image of the centre of the object lies upon the retina, it is clear that, all the other circumstances remaining the same, the image of an object further than PQ would be formed in front of the retina, and that of an object nearer than PQ would be formed behind the retina, by (1) in Art. 65. But in some manner not at present determined, the relative circumstances of the lens and retina can be altered at will within certain limits and

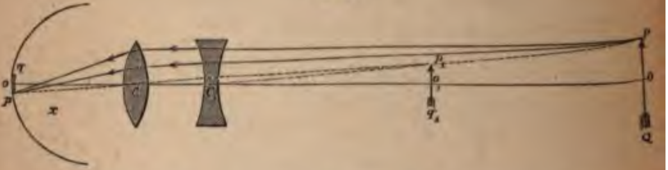
Fig. 52.



with very great rapidity, so that in any case the image is thrown upon the retina.

72. When, in spite of this power of self-adaptation, the image of a distant object is formed in front of the retina as at x in the figure 53, such an eye is said to have *short sight*, which is occasioned by the too great convexity of the lens. It is clear that if by means of another lens we are able to counteract this convexity, we shall rectify this defect, and a little consideration will shew that a double *concave* lens is the best to use, since it performs the required office when held close to the eye, and forms an upright image, while a double *convex* lens, although it would bring the object nearer as required, yet, since it forms its image between the eye and the lens, would have to be held at an inconvenient distance from the eye, and the image would be inverted.

Fig. 53.



But now suppose that the image of a near object is formed behind the retina. Such an eye is said to have *long sight*, and can have this defect corrected by any lens that will throw the image farther back. Such a lens is a convex lens with a very long focal length, as can be seen by fig. iii. Art. 65.

The eye of a fish is of course adapted to see objects in the water, a dense medium. The lens in the eye must ıre be either very dense or nearly spherical so as to :he rays sufficiently. The latter is the case. Thus, iately a fish is brought out of the water, it is afflicted ery short sight.

'he long-sighted eye receives greater benefit from ex lens than a short-sighted eye does from one that :ave. For an object seen through a convex lens is ıly magnified, but it is seen brighter by the eye ng a larger pencil of light from each visible point, on ıt of the rays entering it less divergent. On the ry, a concave lens not only diminishes objects, but also s them darker, because the rays are thus transmitted liverging to the eye, and consequently the pupil can- :eive so many of these rays as it otherwise would. It this account that short-sighted people cannot see : objects, unless they be very large and bright, so ırough a concave lens, as the theory of images formed ɛ retina may seem to promise. For the principal iments to the distinct vision of remote objects are vant of light and magnitude, and both these defects rther increased by concave lenses." (*Encyclopædia ɔolitana.*)

CHAPTER XI.

Telescopes.

73. Roger Bacon, A. D. 1250, has treated, amc Optical Phenomena, upon the apparent magnitude jects and spherical foci. He says, "For it is easy stand, by the canons above mentioned, that the things may appear exceeding small, and contrar we can give such figures to transparent bodies, anc them in such order, with respect to the eye and the that the rays shall be bent and refracted towards we please; and thus, from an incredible distance, read the smallest letter, and may number the sma ticles of dust and sand, by reason of the *greatne angle* under which we may see them; and, on the we may not be able to see the greatest bodies ju by reason of the smallness of the angle under wh may appear. *For distance does not affect this kind* excepting by accident, *but the angle does.*" Thu really produced effects precisely analogous to those by our telescopes and microscopes.

In 1570, in the first English edition of Eucl ments, by Sir Henry Billingsley, there are mention spective glasses," clearly the same as our telesc 1591, Thomas Digges of Oxford writes thus: "My f his continuall painful practises, assisted with demon mathematicall, was able and sundry times hath, by proportionall glasses, duly situate in convenient angles, not only discovered things farre off, read letters...but also seven miles off declared what hath been doone at that instant in private places."

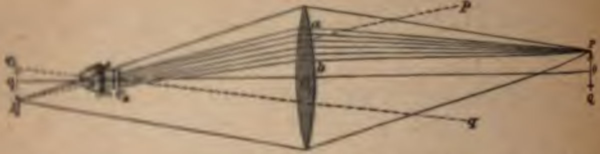
In 1610, so powerful had these glasses been made, Harriot was able to observe the spots in the sun.

Towards the end of the 16th century, a fortunate concurrence led to the invention of dioptric telescopes in Holland. Borelli informs us that the children of Zachariah Jansen, a spectacle-maker of Middle-Burgh, amusing themselves in their father's shop, placed by chance a convex and a concave lens in such a way, that on looking through them at the weathercock of the church, it seemed to them much larger and nearer than usual...This was about 1590. No sooner was this interesting discovery made known...than Galileo, Kepler, Descartes and other philosophers bent the whole force of their genius to the improvement and employment of so useful an apparatus. (*Encycl. Metropol.*)

74. It would be well, before discussing what it really was that these children had found out, to give a slight explanation of the words in italics in Bacon's writing quoted above.

If we hold a small circular piece of paper between us and the sun, we can, by placing it at the proper distance, exactly eclipse the sun: and we say then that the sun and the paper appear the same size. If we did not know the distance of the paper or its size, the eye would at once convey to the brain the idea that the sun and paper were of the same size. In a total eclipse, for instance, of the sun by the moon, this occurs. The moon though much smaller than the sun, appears to cover the sun, that is, appears to be the same size. Thus size is judged by the relative *angles*, not by the relative lengths and breadths. It will be necessary constantly to bear this in mind in what follows.

Fig. 54.



75. Figure 54 will shew how it was that the children of Jansen saw the weathercock larger, by means of a convex (C_1) and a concave (C_2) lens.

Suppose for the moment that C_2 was removed. The rays from P of the weathercock would, after refraction at C_1, form an image at p_1, where PCp_1 is a straight line, and $\frac{1}{C_1p_1} - \frac{1}{C_1P} = \frac{1}{f}$, where the quantities have their proper signs, according to the usual convention. Similarly Q would have its image at q_1. Now let the concave lens C_2 be placed so that p_1q_1 is just in its focus. The rays then which are coming from the object-glass to the focus p_1, and are intercepted by the lens C_2, will on emergence from C_2 form a beam of rays parallel to p_1C_2. But when rays enter the eye in a parallel beam, they appear as if they came from a point a very long way off. Thus, then, these rays appear to come from some point p (i.e. the eye sees an image) a very long way off, in the direction p_1C_2 produced. Similarly with q. So that, finally, the object PQ, by means of the lenses, is seen under an angle pC_2q instead of under the angle PEQ or PC_2Q, as it would be without the lenses.

We have in the last few words assumed that the eye and eye-lens coincide. In order to gain the central rays of : pencil falling on the eye-lens from each point of the iect, it is clear that the eye must be placed close to the s. Whence it follows that there is no object in having eye-lens much larger than the pupil of the eye. Thus, the figure, only a very small portion of the rays falling on : object-glass come to the eye.

We have therefore now shewn how an image is formed *ger* than PQ. But the children saw it also *nearer*, where- by our method, it has been placed at an infinite dis- ıce. A very slight alteration in the position of the lenses l accomplish this. Suppose we push in our concave lens y slightly, nearer to C_1. The result of this, since an ject and image move in the same direction, will be to e the emergent rays a slight divergence from some point m p_1C_2 produced, not at an infinite distance, as before. ıe *size* of the final image pq remains the same as before, ce it is measured by the angle under which it is seen, d that is not altered, except by the very slight movement C_2.

For objects so distant as the moon, for example, or leed for objects much closer, at the distance of a few ıdred yards, the rays incident on the object-glass may be ısidered parallel, and p_1q_1 is therefore in the principal us. Therefore C_1C_2 = difference of focal lengths of the lenses.

76. This combination of lenses is called Galileo's scope, from the following facts. "Happening to be in ıice in the month of May 1609, he learned in that city : a Belgian had invented a perspective instrument, by ıns of which distant objects appeared nearer and larger.

This report...excited an intense interest in his upon his return to Padua he began to ponder probable mode by which such an effect could be He discovered upon the very night of his arrival, enlargement of the object depended upon the d refraction, and on the following day he made attempt to execute an instrument upon this Having procured a leaden tube, he fitted in o extremities a plano-convex lens, and in the other concave one, and, having applied his eye to the lens, he was delighted to perceive objects pre magnified. They appeared to be three times ne nine times larger....He immediately gave intimat success to his friends in Venice, with whom he conversing on the previous day. Having succeed days later in making a better instrument, he joy ceeded to Venice, taking it with him....Galileo's o was to construct a telescope of superior magnifyin This he found to be a task of very difficult accomp for the art of grinding and polishing lenses was th infancy....Soon afterwards he made another i which magnified 60 times in surface, and finally, neither application nor expense, he succeeded in an instrument of such excellence as to represen almost 1000 times larger, and above 30 times ne they appeared to be by the natural power of (Grant's *Hist. of Astron.* Vol I. p. 520.)

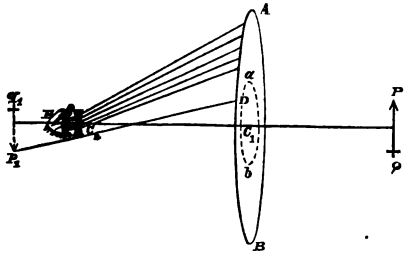
Thus Galileo constructed this telescope *by* thinking *how* it could be done, whereas all former tions had been entirely experimental.

77. There are two great disadvantages atten construction.

(i) The image q_1p_1 is *virtual.* The disadvantage of this will be best seen when we discuss the Astronomical Telescope.

(ii) The small number of rays falling on the object-glass which come to the eye lens. This will be best seen by tracing the rays backwards from the eye to the part of the object-glass from which they have come.

Fig. 55.



Suppose then that the eye of perfect sight receives a beam of rays which have originally come from P. They are, of course, after emergence from the eye-lens, parallel to p_1C_2. Before entering the eye-lens these rays have formed a converging beam from the object-glass. Thus if the eye is to receive such a beam as we have drawn, the object-glass must be of the size AB, much larger than it is possible to make it. Even then, no rays from P which fall on the object-glass below D would come to the eye. Thus with an object-glass as ab, hardly any rays, starting from P, would come to the eye, so that scarcely any image of P would be seen. In the Astronomical Telescope, as will be seen, usually all the rays which fall on the object-glass fall also on the eye-lens and are of use.

78. The proportion of the angle which the image by the eye subtends at the eye to the angle which the o would subtend at the eye, without the lenses, is calle "magnifying power" of the telescope.

Thus in fig. 53 "magnifying power" $= \dfrac{pEq}{PEQ}$.

Since the eye is close to the lens, we may take stead of E.

Thus

$$\text{"magnifying power"} = \frac{pC_1q}{PC_2Q} = \frac{p_1C_1q_1}{PC_2Q}$$

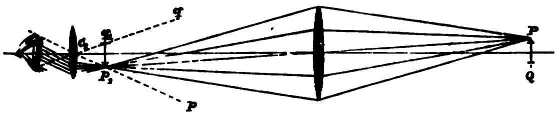
$$= \frac{PC_2}{p_1C_1} \text{ very nearly} = \frac{\text{focal length of O. G.}}{\text{focal length of E. G.}}.$$

Galileo points out a simple method of determinin experiment the magnifying power of the instrument. "Pl says he, "upon a wall at a certain distance two disks of which you will observe with the telescope, and one the naked eye. If the disk seen through the teles appear equal to the external one, the magnifying pow in the proportion of the two disks. If they do not ap equal, enlarge or diminish the external disk until this takes place, and then the magnifying power will b mentioned, in the proportion of the two disks" (Gr *Hist. of Astronomy*).

79. Two years after Galileo's discovery of this scope, Kepler first explained the construction of a teles formed by the combination of two convex lenses, He did not, however, manufacture one himself. In Scheiner made observations in the presence of the duke Maximilian with such an instrument.

In this case the eye-lens is placed so that the image $_1$ is between it and the object-glass, and is in its focus. ne rays then which have come from P, after emergence m the eye-lens, come out in a beam of rays parallel to

Fig. 56.



$_1C_2$. That is, the eye behind C_2 sees an image p of the oint P at an infinite distance in the direction C_2p_1 prouced. Similarly with Q.

As before, the rays from P to the object-glass may be upposed parallel, so that p_1q_1 is in the focus of C_1. There-re $C_1C_2 =$ *sum* of focal lengths. By pushing our eye-lens lightly nearer to p_1q_1, we could give the emergent rays slight divergence, as shewn in the Galilean telescope, and ius make them appear to come from a point p *not* at an ifinite distance.

80. Now we have to remark that here we have made, r convenience of drawing, our eye-lens much larger rela-vely to the object-glass than it really is in practice. This as done for the sake of taking in the whole pencil. But we have a very small eye-piece in the same position we iould lose some of the rays from p_1 for instance. This e could obviate partly by taking an object-glass c_2 with a eater focal distance, since then the rays from it would nverge to and diverge from p_1 more slowly, and so the e-piece would take more of them in. And the greater e focal length, the more would this be the case. ("Thus uyghens states that he and his brother made object-glasses

of 170 and 210 feet focal length. These glasses were used without tubes. He presented one, having a focal length of 123 feet, to the Royal Society of London. While he was engaged in these labours the art of executing telescopes was attaining a high degree of excellence in Italy, but they were constructed chiefly in France and England. Auzout made one of 300 feet focal length.") If, again, we had an eye-piece with a very small focal distance, we should thus catch more rays, since the eye-piece would be closer to the image, but we should in that way have a greater magnifying power than is convenient.

We saw in Art. 76, that Galileo fitted his lenses at the ends of a tube. The use of this tube was to shut out the interference of all light that was not actually being used, *i.e.* all except the rays from the object on the object-glass. Thus, what a refracting telescope does is this. It collects a great many rays from a point of the object and sends them to the eye in a dense small beam of parallel rays (whereas the eye without the lenses would only receive the rays that fell on its pupil), thus giving to the eye an image of the point at an infinite distance. If the eye is not long-sighted enough for parallel beams, by shoving the eye-piece slightly in, we can, as said in Arts. 75, 79, give the rays the required amount of divergence.

81. Comparing the Galilean and astronomical telescopes we notice the following differences.

(1) In the Galilean the distance between the lenses equals the *difference*, in the astronomical it equals the *sum*, of the focal lengths.

The Galilean construction is therefore useful for opera-glasses, &c. which require to be short.

(2) In the Galilean only a small portion of the rays which fall on the object-glass are of use. In the astronomical usually nearly all are of use. See figs. 55, 56.

Thus, in fig. 55, if a piece of paper were placed in front of the portion *ab* of the object-glass, no image of *P* would be seen at all, since it would cover the only part of the object-glass by which *P* is seen. On the other hand, if any portion of the object-glass of the astronomical in (56) were covered, the only effect would be to dim the brightness of the whole.

(3) In the Galilean the image is *erect:* in the astronomical *inverted.*

(4) In the Galilean the image is *virtual,* in the astronomical it is *real.* This advantage of the astronomical telescope and (2) quite put out of sight the comparatively small advantages (1) and (3) possessed by the Galilean.

"The first person who appears to have discovered the peculiar value of the telescope recommended by Kepler, and who applied it to the purposes of astronomical investigation, was Gascoigne. He did not fail to perceive that it possessed an immense advantage over the Galilean telescope in affording a situation within the tube where any object being placed might be viewed as distinctly as a celestial object. This suggested to his inventive mind the use of telescopic sights in astronomical observations, and the application of the micrometer to the telescope." If, for instance, a series of wires were placed inside the telescope at the focus where p_1q_1 is formed, the eye would see both wires and image, and the relative breadths, for instance, of two planets might be found by noticing how many wires they respectively covered; or a ring might be placed there to limit the amount of the image that the eye was to see.

What we have called 'wires' in telescopes are in reality threads of a spider's web.

(5) In the Galilean the eye should be placed close to the lens. In the astronomical it should be at E when the central rays of the pencils from the extreme points of the object cut the axis, since otherwise it would lose a great many of the rays.

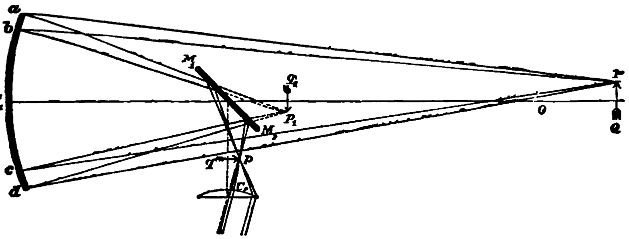
82. The great length necessary in these telescopes made it desirable to discover some new construction. In a letter written to Descartes in 1639 Mersenne suggested the idea of a reflecting telescope. Descartes offered several objections to the proposal, and it was not carried into effect. It is impossible to ascertain whether Gregory was indebted to any previous writer for the hint of the reflecting telescope or not: but, at any rate, he was the first person who gave an explanation of it.

The first person who actually executed a reflecting telescope was Newton. But the difficulties in his way were very great, since every irregularity in a reflecting surface would make the rays deviate five or six times more out of their due course than in a refracting telescope. While his mind was occupied with these reflexions, he was compelled by the plague to leave Cambridge, and more than two years elapsed before he again considered the subject. Accordingly, having succeeded, after much trouble, in discovering a substance adapted for the formation of his larger mirror, and having also devised a delicate mode of polishing the surface, he actually executed a telescope upon this principle. The original idea had been to make a parabolic mirror, which, as we saw (Art. 15), reflects parallel rays accurately to the focus. Newton however wisely abstained from attempting to make it truly parabolic, on account of the extreme difficulty of the operation, contenting himself with giving it the figure of a segment of a sphere. With this instrument he could see the satellites of Jupiter and the horns of Venus. Encouraged by this success, he made a second telescope, still better than the first. The focal length of the speculum was $6\frac{1}{3}$ inches, and that of the eye-glass (which was a plano-convex lens—the flat side turned towards the eye) was $\frac{1}{6}$ of an inch. The magnifying power was therefore 38. See Art. 78.

In December, 1671, Newton sent this telescope to London, to the Royal Society, from whom it received unanimous commendation. A description of it was sent to Huyghens in Paris, by whom it was communicated to the Academy of Sciences, and in consequence it was soon generally known on the continent. (Grant's *Hist.* p. 528.)

83. We will now give a sketch of the telescope devised by Newton.

Fig. 57.



C_1 is the mirror which Newton made in the shape of the segment of a sphere instead of that of a parabola. We will suppose that PQ is so far off, that the rays Pa, Pb are to be considered parallel. Then an image of PQ would be formed at p_1q_1 where C_1p_1 and C_1q_1 equal $\frac{1}{2}$ radius of C_1 (Art. 22). PQ should, of course, be much further off, so that p_1OP, q_1OQ may be straight lines. But Newton placed

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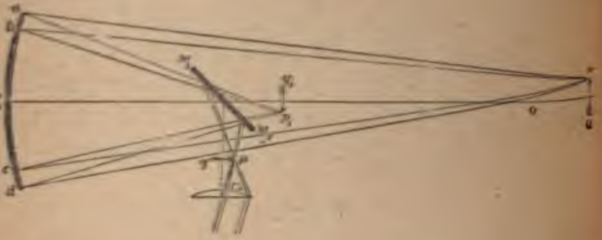
a plane mirror M_1M_2 inclined at an angle of 45° to the axis the effect of which is that the rays coming to p_1 are reflected off to p, and those from q_1 to q, where the lengths of the reflected rays just equal what they would be if there were no mirror (Art. 6). And $pq = p_1q_1$ and pq is at right angles to p_1q_1. The plano-convex lens C_2 is placed with its plane parallel to the axis so that pq is exactly in its focus (supposing the sight to be perfect sight), whence the rays which fall from p emerge in a parallel beam, and the image is seen at an infinite distance in the direction pC_2.



84. An object must be to make M_1M_2 as small as possible, since it intercepts a great many rays and so prevents them from coming to C_1. If we join q_1 to the top of the mirror and p_1 to the bottom, those lines form the boundary of the reflected rays from PQ and therefore of the plane mirror. If we make it larger, we shall cut off some rays from PQ from reaching C_1, and gain no recompensing advantage. If we make it smaller, more rays will fall on C_1, but the extreme rays will not come to M_1M_2.

85. From the figure we see that the object is seen at right angles to its proper direction. Two other kinds of reflecting telescopes were invented, one before Newton's, by Gregory (not however manufactured till afterwards, by

Iooke, 1674), and the other by Cassegrain, a Frenchman, in 672. The advantages of these were that the image was iewed directly instead of at right angles.

In 1723 Hadley presented to the Royal Society a eflecting telescope of the Newtonian form, the diameter of he large mirror being six inches, and its focal length five eet three inches. When compared with the refracting elescope of 123 feet focal length, presented to the Society y Huyghens, it was found that it bore as high a magnifying ower as that instrument, and exhibited objects equally istinct, though not so bright.

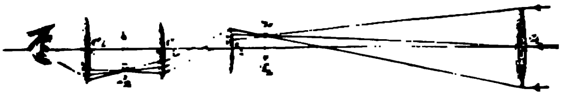
As noticed in other telescopes there is great advantage 1 having a mirror with a long focal distance, since the mall plane mirror will then catch so many more of the rays eflected off the big mirror.

"The telescopes which Herschel employed in the early art of his astronomical observations were all of the New-onian construction. In 1786, however, he laid aside the mall mirror, and instead gave a slight inclination to the irge mirror, so as to throw the image a little to one side of he tube, and thus make it possible to view it directly with n eye-glass. By this contrivance the light usually absorbed y the small mirror was saved, and the illumination of the nage increased in a corresponding degree. In such a tele-cope it is obvious that the observer looks at the image ith his back turned to the object. In 1789 Herschel urpassed all his former efforts by the completion of a elescope forty feet focal length and four feet aperture. It s impossible, within a moderate compass, to give even simple enumeration of the multitude of brilliant dis-overies in celestial physics which rewarded the labours of his great astronomer."

The advantages of these reflectors obviously consist in their being much shorter than refractors of equal power, and in the facility with which different eye-pieces may be applied to the same speculum. Among their disadvantages are the difficulty of directing them to the objects, but particularly the extraordinary precautions that are necessary as well in the application of them as in their construction: they are also very expensive, in consequence of the metal speculums, and are easily put out of order.

86. We stated in Art. 81 that a slight advantage of the Galilean telescope over the astronomical was to be found in the fact that the image seen in the former was erect, while in the latter it was inverted. We give a sketch of an eye-piece which might be used to counteract this defect in the Astronomical Telescope, not from any practical knowledge that might be gained by it, since it has long been superseded by improvements, but because it gives an useful exercise in tracing the course of a small pencil.

Fig. 58.



p_1 is just in the focus of C_2, therefore the rays w
emerge parallel to p_1C_2.

After refraction at C_3 the parallel rays will be b
to the focus p_2 of C_3.

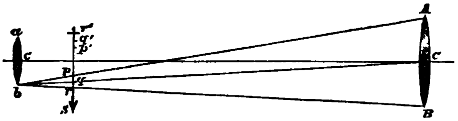
If p_2 is just in the focus of C_4, the rays will em
parallel to C_4p_2.

Thus, finally, the object p is seen in direction
instead of C_1p.

87. In Art. 81 (4) it was mentioned as one of the
vantages which the astronomical telescope pos

ver the Galilean, that from the fact of the image being *real* nstead of virtual, it was possible to limit the amount which he eye wished to see, by placing a ring in the focus where he image was formed. We will give some further explanaion of this.

Fig. 59.



Draw straight lines from *b* the bottom of the eye-lens to *A*, *C* and *B*, the top, centre, and bottom of the object-glass, and let them cut the image in the points *p*, *q*, *r*.

Suppose now that a ray came from *A* to any point of the image below *p*. It will pass *below* the eye-lens, and the eye will not get the benefit of it. Thus the image will be slightly dimmer just below *p*. And as we look lower and lower at the image this dimness will increase, since more and more rays will pass underneath the eye-lens. And when we come to *q*, just half the rays are of use, viz. those falling on the lower half of the object-glass. Then *c* is said to be seen by half pencils. Below *q* the image will be seen by less than half pencils, less and less, and *r* is the last point that will be seen at all, since of all the rays converging from *ACB* to any point *s* below *r*, not one comes to the eye-lens. Similarly, no point above *p'* will be seen quite clearly. This indistinct piece in the image is called the "*Ragged Edge:*" it is found convenient to cut it off from sight, which is done by the ring in the focus, the aperture of which equals *pp'*, so as just to allow the whole of the distinct portion of the image to be seen. This ring is called the "*Stop.*"

This could not be done in the Galilean telescop the image does not really exist.

The size of this "Ragged Edge" can be found th

$$\frac{pr}{AB}=\frac{bq}{bc},$$

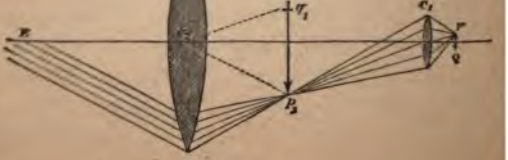
$$\text{or } \frac{\text{Ragged Edge}}{\text{diam. of object-glass}}=\frac{\text{focal length of eye-glas}}{\text{f. l. of eye-glass + f. l. of obj.}}$$

88. There is one more instrument which we w scribe, as it differs in its object from the telescope discussed, although in construction it is very simil the *microscope*. We shall give its construction in its si form.

The object of the instruments already treated on magnify and view clearly very distant objects; the therefore called "*Telescopes*." The rays on the objec might then be regarded as parallel, and after p through it, converged to the principal focus of the glass.

The object of the present instrument is to magni view clearly very minute objects at a very small dis so that the rays can be no longer regarded as paral the object-glass. The manner in which this is done seen by the following figure.

Fig. 60.



C_1 is a convex lens with a very small focal length. PQ is a minute object placed slightly further from C_1 than its focus. If PQ were exactly in its focus the emergent rays would be parallel to PC_1. Therefore since object and image move in same direction, the rays from P will converge to a point p_1 on PC_1 produced, where

$$\frac{1}{C_1p_1} - \frac{1}{C_1P} = \frac{1}{f}.$$

These rays are then viewed as in the Astronomical Telescope by another convex lens placed so that p_1 is in its focus. The rays then emerge parallel to p_1C_2. The image therefore is seen under the angle $p_1C_2q_1$, whereas without the lenses it would be seen under the angle PEQ.

The smaller the focal length of C_1, as also of C_2, the greater would be the angle $p_1C_2q_1$, *i. e.* the greater would be the magnifying power. Object-glasses are now made for microscopes with focal length of not more than $\frac{1}{50}$th inch. From the fig. it is plain that the eye-lens must be considerably larger than the object-glass.

To explain the fact that a star seen indistinctly through a telescope on account of the brightness of the night, can be made clearly visible by employing a higher magnifying power, while a planet cannot. If a faint star be viewed on a bright evening, it may be invisible on account of the brightness of the field of view. If now we increase the magnifying power, we do not increase the total quantity of light that enters the telescope, because that only depends upon the size of the object-glass. But by increasing the magnifying power we spread the same light over a larger area, and therefore the field of view appears fainter. If the star be a planet it will have a sensible disk, which will be magnified in the same proportion as the field of view, an the relative brightness will therefore remain unalter But in the case of a star there will be *no* sensible disk, an the brightness of the star will therefore be unaltered, th is, will appear relatively brighter than the field of view.

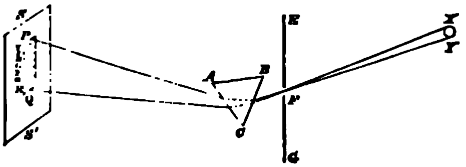
CHAPTER XII.

Chromatic Dispersion.

89. We have, up to the present point, assumed that whatever happens to a ray of light, whether it be reflected or refracted, its *quality* remains always the same; that it is like a ball, which, after being thrown against a wall, is exactly the same ball as before it struck the wall. It was not until many attempts had been made to explain the Rainbow, that Newton stated, in 1675, the theorems that "Lights which differ in colour differ in refrangibility," and "The light of the sun consists of rays differently refrangible." The latter of these is the one with which we shall be chiefly concerned, but Newton's first experiment to prove the first of them is so simple and instructive, that we give it.

He took a black oblong stiff paper terminated by parallel sides, and painted the upper half red and the lower half blue, and viewed it through a prism of a refracting angle of 60° held parallel to the sides of the paper and the cross line. He then found that if the refracting angle of the prism were turned upwards, so that the paper might seem to be lifted up by the refraction, the blue half was lifted higher than the red half: and if downwards, the blue half was carried lower than the red half: which shewed that in both cases the light from the blue half of the paper suffered a greater refraction at the prism than that from the red half. We will now state the experiment by which Newton proved the second of the theorems above, viz. that "Light of the sun consists of rays differently refrangible."

Fig. 61.



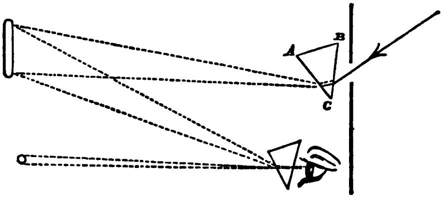
He allowed the sun to shine through a round hole *F* in a shutter, about $\frac{1}{4}$ of an inch in breadth. He then placed a prism *ACB* behind the hole, so that for the very small pencil from *F* falling upon it it should be in the position of minimum deviation for convenience, which position he settled in the way given in (Art. 56). Some distance behind the prism he placed a screen *SS'* to receive the refracted rays. The image *PQ* formed was oblong and not oval, but terminated with two rectilineal and parallel sides by which it was bounded pretty distinctly, while its ends were confused and indistinct, the light there decaying and vanishing by degrees. The *breadth* of this image answered to the Sun's diameter. This elongated stripe *PQ*, the height of which is about five times its breadth, is found to be of different colours, these colours being ranged in the following order: violet, indigo, blue, green, yellow, orange, red.

90. This experiment then shews that a ray of sunlight is not one colour by itself, but is made up of different colours, between which the eye does not distinguish, until they are detected for it by the new medium, which does not act on one colour in the same way as on another.

It is easy to see why the colours in this stripe, which is called the solar "*spectrum*," are not pure, *i.e.* appear to run into one another. Each separate ray of the small pencil from *F* has its own spectrum: thus the yellow, orange, and red of the spectrum of the uppermost ray would come upon the blue, green, and yellow of the middle ray, and upon the violet, indigo and blue of the lower ray, and thus no colours would be perfectly pure, except the extreme edges of the violet and red.

91. Newton also shewed that all the various portions of this coloured band, when again brought together, produce upon the eye the effect of white light.

Fig. 62.

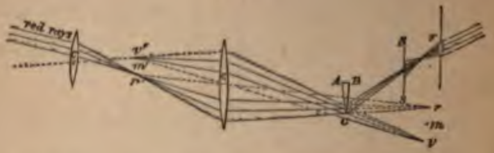


This experiment he performed by simply allowing the light passing through the round hole *F* to fall on *ABC*, producing the solar spectrum, and then, on looking at this coloured band by another prism (*abc*) placed in the same direction, instead of seeing a coloured band, he observed a spot of white light. (Roscoe.)

92. We stated in Art. 90, that the spectrum was not quite pure. But in order to examine the spectrum usefully, it is necessary to have it quite pure. This object we can attain by the following means.

(i) Let a small convex lens *F* of small focal length be placed in the hole in the shutter. An image of the Sun will be formed in its principal focus, since the focal length is very small the image be very small, so small indeed that we may re it as a mere luminous point.

Fig. 63.



(ii) Let a very small pencil from this image of the fall upon the prism ACB, which is placed in position of minimum deviation for mean rays, for rays of a mean amount of refrangibility, remainder of the rays from the image being cut by SS'. This prism refracts the violet rays mo and the red rays least. We will consider only tho two colours for simplicity. Since our prism placed in the position of minimum deviation, th red rays will appear very nearly to diverge afte emergence from a point r, and similarly for viole (Art. 57).

(iii) Let the rays now be received on the convex len C_1.

The red rays will then converge to r', where rC_1 is a straight line, and $\frac{1}{C_1r'} - \frac{1}{C_1r} = (\mu_r - 1)\left(\frac{1}{r} - \frac{1}{s}\right)$ similarly for the violet, μ_r being the index of re fraction for red rays.

The spectrum $r'v'$ is pure, but is too small to b observed, since clearly it is hardly more than mere *points* of colour, therefore

(iv) Lastly, let them be received upon another convex lens C_2, so placed that m', the point to which the mean rays converge, is in the principal focus. The rays from r' will then emerge parallel to $r'C_2$, those from v' parallel to $v'C_2$, and we can by placing our eye at the proper place behind the lens C_2 view the image of the spectrum $r'v'$ directly. This is obviously using the simple astronomical telescope to view the object rv.

93. The great practical difficulty in performing this experiment when tried with sunlight, consists in the fact that as the sun moves, the light does not fall in the same direction on F, and its image therefore moves. To obviate this we must have, outside the shutter, an instrument called a "heliostat." This is a mirror revolving about an axis parallel to the Earth's axis with one-half the sun's angular velocity, so that it shall always reflect the sun's rays in a fixed direction, viz. on to the lens.

94. It can be shewn that the spectrum thus found cannot be any further broken up. If we take for instance only the red rays in the last figure, after emergence at C_2, and let them fall again on a prism, their colour is unchanged at emergence. Thus, finally, the foregoing experiments shew us that ordinary light is made up of a number of rays of different refractive indices varying between close limits.

95. In Art. 91, we shewed how Newton proved that all the various portions of this coloured band, when again brought together, produce upon the eye the effect of white light. The same thing can be proved, or rather illustrated, by a very easy experiment, which depends upon the fact

that light remains a short time on the eye. This f perhaps be shewn most simply thus. Place a whit at the end of a string, and whirl it round. A the eye is able to follow the bead round and rour when it attains a certain pace, the eye sees only a ring. This then shews that the light which came to t when the bead was in any fixed point of its orbit mained on the eye during the time that the bead ta make one revolution. This time is somewhere abou of a second.

If now we take a disk of cardboard, and paint sectors with the colours of the spectrum in their proportion, we shall, if we place it on a top rev 15 times per second, gain the same effect as if the c were mixed together in the same proportion, that shall see a *white* disk.

Now the reason why all these different coloured when they reach the eye, produce the effect of white is a physiological question which we cannot explain. find that not only do all the colours of the spectrum, they are brought upon the eye, produce the effect of light, but that several mixtures of only a few out of all different colours have the same power of producing the eye the effect of white light: for instance, viole greenish yellow; indigo and yellow; blue and or greenish blue and red.

96. There is a curious analogy between the h eye and ear which this spectrum brings out. It is known that there are certain sounds which, and higher which, many ears cannot hear at all, such as a grassho chirp. The sensation of sound is produced by the tions of air acting on the nerves of the ear, each s nerves being acted upon by one sound. If then the r

on which such a sound as that mentioned would act are inting, the ear would not hear that sound.

Now in the same way it is known that the refracted ys do not end with the visible part of the spectrum. his is found from the fact that outside the spectrum the ating power can be felt although there is no light. Indeed e maximum heating effect is obtained at a point outside e red portion of the spectrum. The blue and violet ortion of the spectrum is that least endowed with heating ower, but this portion and the invisible portion extending o some distance beyond the violet end are of very great ervice from the fact that they produce chemical action, such as the decomposition and blackening of silver salts, upon which the art of photography is based.

97. The angle between any one of the coloured rays and the white ray produced is called the "*deviation*" of that coloured ray.

The difference between the deviations of two colours is called their "*dispersion*," and the difference between the deviations of the extreme colours is called the "*dispersion of the pencil.*"

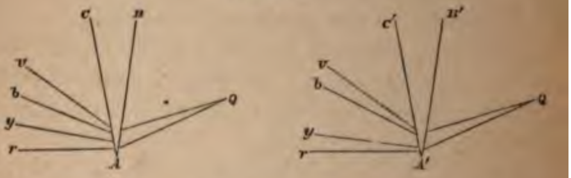
In treating on prisms we shewed (Art. 55) how to find the value of the refractive index μ for any medium. Now if we find in this way the values of μ for all the colours of the spectrum in several different media, we discover the fact that the ratio of the dispersion of any two colours to the dispersion of the extreme colours of the spectrum *is not constant* when different media are used.

Thus, if ABC, $A'B'C'$ be two different media, and the angle between the violet rays and red rays, v and r, is the same in both, it is found that the angle between b and the blue and yellow rays, is not the same.

This is called the "*Irrationality of dispersion.*"

98. Hence, if two prisms can be made so as to give t same *total* dispersion, and if they be placed so as to refra

Fig. 65.



in opposite directions; in the emergent beam the extren red and violet will be united, but not the intermedia colours completely.

Since our prism is supposed placed in the position minimum deviation, if μ_x, μ_y be the indices for any tw colours and D_x, D_y their deviations, the angle of the pris being small, then $D_x = (\mu_x - 1)A$, $D_y = (\mu_y - 1)A$; $\therefore D_x - D_y$ $(\mu_x - \mu_y)A$, Art. 53; and therefore the dispersion varies as but besides this it varies as $\mu_x - \mu_y$, that is, (i) "the *sam* substance disperses *different* colours differently." Agai if we take several prisms with the same refractive angl but made of different media, the dispersion for two fixe colours is not always the same. Thus (ii) "*different* sub stances disperse the *same* ray differently."

99. It is found also that the dispersion of any tw colours is not proportional to the deviation of the axis the pencil. This is clearly a most fortunate circumstanc If it *were* proportional, another way of saying it would b

that any combination of prisms or lenses which destroyed dispersion must destroy deviation too. Now the fact that a lens or a series of lenses magnifies an object arises simply from the fact that the direction of each ray suffers deviation, and the greater this deviation, the more is the object magnified. Thus then we should, by destroying dispersion, gain a perfectly colourless image, but in no degree magnified.

Since however dispersion is *not* proportional to deviation, media can be found which produce *the* SAME DISPERSION *in opposite directions of a given colour relatively to another*, but DIFFERENT DEVIATIONS in opposite directions in the axis of the pencil. If then a pencil be refracted through these media the two colours in question could be united, while the axis of the pencil suffers a deviation equal to the difference of the deviations which these media would produce separately: but, as we said, the other colours would not be quite united, though nearly. (Parkinson's *Optics*, Art. 173.)

100. Let A and A' be the refracting angles of two prisms of different media, and let μ and μ' be the indices in them for one particular colour. The deviations which the prisms would separately produce for this colour are $(\mu-1)A$ and $(\mu'-1)A'$, when the angles A, A' are small, and total deviation $=(\mu-1)A+(\mu'-1)A'$. If $\mu+x$ and $\mu'+x'$ be the indices for another colour, the total deviation for this colour would be $(\mu+x-1)A+(\mu'+x'-1)A'$. If these deviations be the same and opposite, i.e. if the colours are to be united at emergence from the second prism, then $xA+x'A'=0$, or $\frac{A}{A'}=-\frac{x'}{x}$, (i).

We have thus obtained one relation between A and A'. If now we have another given, for instance that the whole deviation in a given colour is to be a given amount, then $(\mu - 1)A + (\mu' - 1)A' = \text{given quantity}$, (ii), and from equations (i) and (ii) we can find A and A' and thus make our prisms such that the colours considered may be united and the required deviation in a given colour produced. The *minus* sign in (i) shews that the edges of the prism are to be turned in different directions.

CHAPTER XIII.

THE RAINBOW.

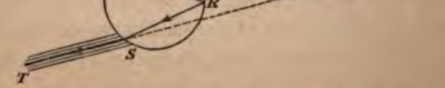
1. EARLY attempts were made to account for the
rs of the rainbow, and various other phenomena in
colours are seen to arise from transient and unsub-
al combinations of media. Thus Aristotle explains
lours of the rainbow by supposing that it is light seen
gh a dark medium: "Now," says he, "the bright seen
gh the dark appears red, as, for instance, the fire of
wood seen through the smoke, and the sun through
Also, the weaker is the light, or the visual power,
earer the colour approaches to the black; becom-
rst red, then green, then purple. But the vision is
gest in the outer circle, because the periphery is
r; thus we shall have a gradation from red, through
, to purple, in passing from the outer to the inner
." The same doctrine is found in the work of De
nis. According to him light is white; but if we mix
the light something dark, the colours arise—first red,
green, then blue or violet. He applies this to explain
olours of the rainbow, by means of the consideration
of the rays which come to the eye from the globes of
, some go through a larger thickness of the globe than
s, whence he obtains the gradation of colours.

escartes came far nearer the true philosophy of the
l colours. He found that a similar series of colours

were produced by refraction of light bounded by shade, through a prism. "At first," he says, "I doubted whether the iridal colours were produced in the same way as those in the prism; but at last, taking my pen and carefully calculating the course of the rays which fall on each part of the drop, *I found that many more come out at an angle of forty-one degrees, than either at a greater or less angle.* So that there is a bright bow terminated by a shade; and hence the colours are the same as those produced through a prism."

102. At length, in 1672, Newton gave the true explanation of the facts. His discovery consisted in teaching distinctly that the law of refraction was to be applied, not to the beam of light in general, but to the colours in particular. The whole explanation of the rainbow turns on this fact and that of "*minimum deviation.*" It was shewn that if a pencil of rays fall upon the face of a prism, that ray which after refraction at the first surface passed through the prism symmetrically with respect to the sides of the prism, was least refracted, and that, of the other rays, those which came out nearest to this ray came out closest together; so that, if an eye received the refracted rays, it would see most light in the direction of minimum deviation (Art. 62). Now the same thing will happen in the case of a sphere instead of a prism.

Fig. 66.

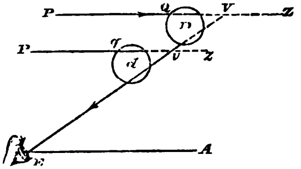


Suppose $PQRS$ a ray falling on a sphere: it is refracted and falls on the further surface at R; part of this ray fracted at R and never meets the eye; but the rest is cted at R, falls on the lower surface at S, and is then, great part, refracted to T.

f now we suppose a great number of rays to fall paral- n the upper side of the sphere, they will come out at lower side, but more will come out very nearly in the ction of the ray which has minimum deviation than in other direction. If $PQRST$ be this ray, the emergent close to ST will form a very small pencil of nearly llel rays, where the angle TVZ or deviation = about . An eye therefore that was placed in a position to ive the rays about TS would see much stronger light this drop than an eye elsewhere. We are now in a dition to explain the rainbow generally. We have erto in this Art. supposed that the light is unaltered by g refracted: but, on entering at Q, it is split up into he colours of the spectrum.

103. (1) Consider light to be of only one colour, say We have then a beam of parallel red rays from the falling on the drop D. The emergent beam will con- of two classes of light, *chief* light and *minor* light, the

Fig. 67.



ef light being that which comes out in the position of imum deviation, and minor light all the rest.

Let PQ be the red ray of minimum deviation: let it, after emergence, fall upon the eye in the direction SE.

Through E draw EA parallel to the incident ray PQ. Then all the spherical drops of rain which lie on the cone whose vertex is E and semi-vertical angle is AED send light in the same state to the eye—i.e. send red light in the direction of minimum deviation.

This is apparent by symmetry. There is no reason, for instance, why the light from one drop D on that cone should be in the least different from that from any other drop d on the same cone.

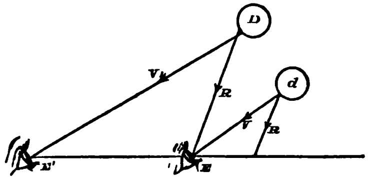
Now recollecting that VE is the ray of minimum deviation, and recollecting that the meaning of that in this case is that rays which fall either higher or lower on the drop than PQ or Pq make on emergence a larger angle EVZ, a little consideration will shew that any drop within that cone will send *minor* red light to the eye. But any drop without the cone will send *no light at all.*

The appearance to the eye would therefore be that of a bright red arch, fading gradually away towards the inside of the arch, but sharply defined on the outside.

(2) Now take lights of all the colours of the spectrum. What has just been said applies equally to each of them; and, since each light has its own direction of minimum deviation, we shall have a concentric series of arches of different colours, EA being their common axis. Since violet is more refracted than red, the minimum deviation of violet is greater than that of red; whence it follows that the red bow is above the violet bow. The other colours would be arranged in their proper order between. The following figure will, perhaps, help to make this point clear.

E be an eye and *D* a drop, such that *DE* is the on of minimum deviation for the red rays, then *E* will 'e the red rays from *D*, but not the violet.

Fig. 68.



'n the other hand it will receive the violet rays from t not the red rays.

'he red bow is therefore seen in the direction *ED*, the violet bow is seen in the direction *Ed*. That e red bow is the outside one.

'he *chief* red light is at the red arch, but minor red as we said, is within the arch, and so for all the ırs. That is, the red part of the bow is the only part .s seen quite pure. Also we noticed that close to the ıow, on the outside of it, there is *no* light; the red bow :refore the most visible, by contrast.

'his rainbow, formed by two refractions and one re- ın, is called the "*Primary*" rainbow.

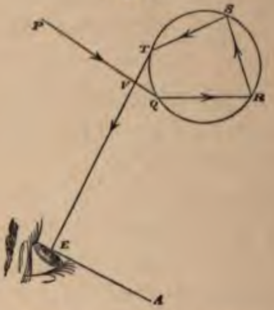
04. The space above the primary bow appears for distance, as we said in the last Article, darker than the Beyond this space appears a broader but fainter ıow, the colours of which are in the reverse order to : in the primary. The origin of this "*Secondary*" bow be explained by pencils which come to the eye after ıg suffered two internal reflexions at the raindrop. kinson's *Optics*, Art. 267.)

At every reflexion and in a less degree at every refraction the pencil loses somewhat of strength. This will account for the secondary bow being fainter than the primary bow, since one more reflexion takes place in it than in the primary bow.

The following figure will serve to shew the course of the rays forming the "*secondary*" rainbow.

The ray PQ is refracted at Q, reflected at R, reflected at S, refracted at T, losing some of its brilliancy at each change. Thus an eye at E sees a faint illumination in the direction ET.

Fig. 69.



The *deviation*, being the angle through which the ray has been bent from its original direction before it meets the eye, is the angle $360^\circ - QVE$. This angle will therefore be greater for violet rays than for red rays, and by reasoning precisely similar to that used in the last Article, the reader can see that the violet arch will in this case be the highest, and the red the lowest; also that the *minor* light of each arc is *above* the arc. Thus the space between the two bows is much darker than that below the primary bow or above the secondary bow.

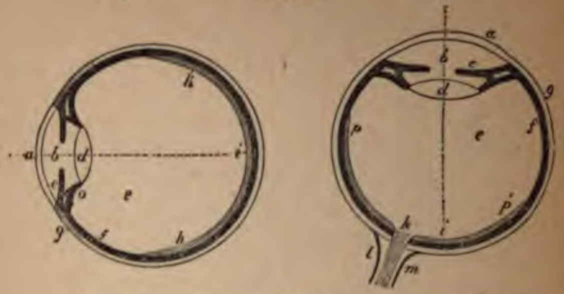
CHAPTER XIV.

THE HUMAN EYE.

105. IN the chapter on lenses we said that for the purposes which we had then in view the eye might be regarded as a double convex lens, forming images of external objects on the retina, which is placed at the back of the eye, and the nerves at the back of which convey impressions to the brain. We will now give a somewhat more full description of the eye.

The following two drawings of the eye, being vertical section and horizontal section of the right eye, seen from above, should be carefully studied.

(i) Fig. 70. (ii)



Vertical section. Horizontal section—right eye seen from above.

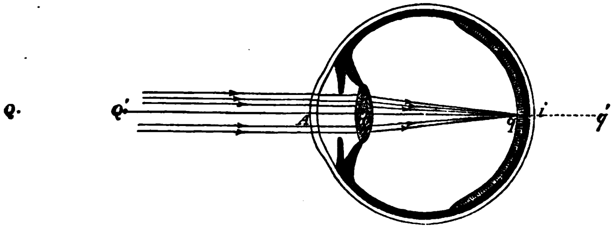
a The cornea, transparent, continuous with *g* th sclerotic coat, opaque.

- *b* The anterior or aqueous chamber, filled with aqu ous fluid, refractive index *less* than in cornea.
- *c* The iris, surrounding the pupil, which is the hole the iris.
- *d* The crystalline lens.
- *e* The posterior or vitreous chamber, containing t semi-fluid vitreous humour.
- *f* The choroid coat of the inside of the eyeball, blac
- *g* The sclerotic coat on the outside.
- *hh'* The retina, network of nerve fibrils to receive imag —vertical section.
- *pp'* The retina, network of nerve fibrils to receive imag —horizontal section.
- *i* The visual centre of the retina, where sight is kee est, also called the "yellow spot."
- *k* The "punctum cœcum," also called the "optic dis
- *l* The optic nerve.
- *m* Continuation of sclerotic, coating the optic nerve.

106. The convex cornea (*a*), receiving, say, parallel rays, causes them to converge: this convergence is slightly diminished at the surface of the aqueous humour (*b*), since its refractive index is less than that of cornea, but again greatly increased at the first surface of the double convex crystalline lens (*d*), and yet again greatly increased at the second surface of the lens, which is considerably denser than the humours before and behind it.

On the whole then we may regard cornea, lens, and humours as a compound object glass, which makes parallel or slightly divergent rays converge to a focus on the retina *hh′*, if the retina is at the proper distance. Now when the eye is perfectly at rest, all its muscles relaxed and no effort at all exerted, the retina *is* at the proper distance to receive the image formed by parallel rays incident on the cornea, is in fact in the principal focus of the compound object glass above mentioned.

Fig. 71.



Thus, suppose the eye is directed to a point *Q* a long way off, so that *Q* is on the *optical axis* of the eye, i.e. the line *Adi*, where *A* and *d* are the centres of the cornea and crystalline lens respectively, and *i* is the point of keenest sight. The rays proceed as described, and converge to *i*, and thus excite on the retina a sensation which is conveyed by the nerve fibrils of the retina to the brain.

107. We have, as we said, here supposed th
to come parallel, i.e. from some very distant object,
moon. But suppose now that the eye is directed to
object a great deal closer, suppose, for instance, we w
pick a hair from the end of a pen: and suppose f
moment the eye to remain in exactly the same st
in the figure above.

Then, since an object Q and its image q move i
same direction, if Q is brought to a point Q', q wou
brought to a point q' *behind* the retina, and the eye
see no distinct image. This, as we said in Art. (72), c
tutes the defect known as "long sight."

If, then, the eye is to see a clear image, i.e. if q' is
on the retina, there must evidently be some change in s
either of the cornea (a) or the crystalline lens (d), o
the distance between the lens (d) and the retina m
some manner or other be altered at will*.

"It has long been disputed what the exact chan
and the dispute is not yet settled. There is a small m
called the ciliary muscle, round the base of the membra
which attaches the lens to the walls of the eyeball
modern writers ascribe to this muscle the function of
ing the shape and position of the lens, but even s
modus operandi remains obscure. A change in the po
of the lens implies great compression of the humour i
or other of the chambers of the eye, and it must b
slight force that shall effect this compression.

I believe it is certain, from direct observation, th
lens is pushed forward in near vision, and also that its
rior surface is made more convex. This latter po

---

* Both of these changes would clearly aid in bringing q' to

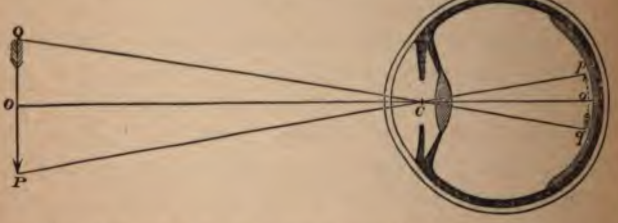
ıed by watching the image of a lamp flame reflected
irst surface of the lens: the image grows smaller
e eye is adapted for near vision. The small ciliary
seems unequal to these effects. More probably they
to the constriction of the eyeball by two strong
called the "oblique," which gird the ball in a plane
angles to its axis. The contraction of this girth
slight elongation to the eyeball in the direction of its
. increases the distance between the lens and the
and therefore fits the latter to receive the receding
f a nearer object. You can feel in your own eyes
withdrawing their gaze from a distant object to an
very near you are performing a muscular exercise;
straining the eyes to the limits of near vision the
r effort becomes very great, and may even be made
ly painful; whereas in re-adapting the eyes to dis-
ion you feel that you are relaxing a muscle; you
strain your eyes for distant vision; the effort is only
rt of abstaining from muscular action, and giving the
time to resume its shape. The limit of distinct
sion is about 6 inches, differing in different people.
yes have an undue convexity of cornea or lens, or an
distance between lens and retina, so that parallel rays
t on the cornea are brought to a focus before they
he retina. This defect is called "short sight," and
be remedied by any effort of the eye, though it is
d by a contraction (instinctive, involuntary) of the
r by almost complete closure of the eyelids, by which
rginal rays of the pencil are excluded, and the mar-
ys are the ones that make confusion and indistinct-
the image. This accounts for the fact of a short-
person habitually compressing his eyelids till there is
chink between them. But the malformation is easily

remedied by the interposition of a double-concave or plano-concave eye-glass in front of the eye, which gives a slight divergence to the parallel rays from a distant object and throws the image of that object further back into the eye till it is within the range of the retina."

"Long-sight," or inability to see clearly objects that are within a foot or two feet, is due to loss of muscular power in the adjustment of the eye for near vision, and is a very common defect in old people. Also in advancing age there is a flattening of the cornea, which throws images a little further back, and makes a greater effort required for bringing the retina into the focus. This is remedied by a convex eye-glass, such as old people generally use.

108. We give a figure shewing the formation of the image of an object of any size, only one point of which can of course lie upon the optical axis. The result is precisely as in a double convex lens, the centre of the eye, that is, the optical centre, lying at about the middle point of the pupil *b*.

Fig. 72.



Thus let *POQ* be an object and let *C* be the optical centre of the compound lens. Then

$$\frac{1}{Cp}-\frac{1}{CP}=\frac{1}{f};\ \frac{1}{Cq}-\frac{1}{CQ}=\frac{1}{f},\ \&c.$$

Range of Vision.

109. When the eye looks straight forward, full face, its ange of vision is much limited; above, by the projecting id and eyebrow; below, by the projecting lid and cheek-bone; inwards, by the projecting nose: but outwards there is no hindrance at all; the most oblique rays from the right meet with no obstacle to prevent their entrance into the right eye; and in like manner the left eye is open to the most oblique rays from the left. Exactly corresponding to this ordinary range of admission of rays from outer objects, is the range of sensitiveness possessed by the retina. This is shewn in figures (i) and (ii). The vertical section shews the range of retinal sensitiveness above and below, and the horizontal section shews the range inward and outward. In the latter it will be noticed that the portion *ih'* of the retina of the right eye is much greater than the portion *ih*, *i* being the visual centre of the retina, in the axis of vision. *ih'* receives rays from the right, unhindered: *ih* receives rays from the left, but the nose being in the way, it does not receive those of great obliquity, and there is therefore no need of retina for such rays. As nearly as I can estimate, the upward range is about 60°, the downward about 50°, the inward about 50°, the outward about 95°. Looking straight forwards, one can detect objects to right or left further back than the horizontal line joining the two eyes.

110. Over the entrance of the optic nerve (Fig. 72 (ii)), there is a circular patch of the retina which is incapable of distinct vision, though not insensible to general impressions of light and colour. Looking with only one eye open, at very large print, it may be noticed that there is an obscuration of a batch of letters about 15° to the outer side of the centre of vision (to the left, looking with the left eye; to the

right, looking with the right eye). This patch of obscurity is about 5° in diameter. Looking with both eyes open, the blindness of each punctum cœcum is corrected by the sharp-sighted tract on the corresponding side of the other retina.

It is pretty well established that the left halves of the two retinæ are supplied with nerve fibres from the left half of the brain, and the right halves from the right. The evidence of this is got partly from dissection, and partly from observation of morbid phenomena: instances are recorded in which disease on one side of the brain was attended with permanent blindness of the corresponding sides of both retinæ. "Half-blindness" of a transient nature, affecting both eyes in the same way, points to the same conclusion. [Read an interesting paper by Dr Woollaston in the *Philosophical Transactions* of the Royal Society, 1824, I. p. 222.]

111. It will be observed that images on the retina are upside down, and we might therefore expect to see everything upside down. But since we see *everything* upside down, this effect is destroyed. "What has the mind to do with upside down so long as it gets evidence of position and motion of things external *in harmony* with the uprightness of the body, and the fact of gravity pulling everything to the earth?"

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APPLIED MECHANICS FOR ENGINEERS

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CIVIL ENGINEERS
NEW YORK.

APPLIED MECHANICS
FOR ENGINEERS

A TEXT-BOOK FOR ENGINEERING STUDENTS

BY

E.dward L.ea HANCOCK

ASSISTANT PROFESSOR OF APPLIED MECHANICS
PURDUE UNIVERSITY

New York
THE MACMILLAN COMPANY
1909

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Norwood Press
J. S. Cushing Co. — Berwick & Smith Co.
Norwood, Mass., U.S.A.

PREFACE

In the preparation of this book the author has had in mind the fact that the student finds much difficulty in seeing the applications of theory to practical problems. For this reason each new principle developed is followed by a number of applications. In many cases these are illustrated, and they all deal with matters that directly concern the engineer. It is believed that problems in mechanics should be practical engineering work. The author has endeavored to follow out this idea in writing the present volume. Accordingly, the title "Applied Mechanics for Engineers" has been given to the book.

The book is intended as a text-book for engineering students of the Junior year. The subject-matter is such as is usually covered by the work of one semester. In some chapters more material is presented than can be used in this time. With this idea in mind, the articles in these chapters have been arranged so that those coming last may be omitted without affecting the continuity of the work. The book contains more problems than can usually be given in any one semester.

While it is difficult to present new material in the matter of principles, much that is new has been introduced in the applications of these principles. The subject of Couples is treated by representing the couples by means of vectors. The author claims that the chapters on Moment of Inertia, Center of Gravity, Work and Energy, Friction and Impact are more complete in theory and applications than those of any other

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American text-book on the same subject. These are matters upon which the engineer frequently needs information; frequent reference is, therefore, given to original sources of information. It is hoped that these chapters will be especially helpful to engineers as well as to students in college, and that they will receive much benefit as a result of looking up the references cited. In general, the answers to the problems have been omitted for the reason that students who are prepared to use this book should be taught to check their results and work independently of any printed answer.

The author wishes to acknowledge the helpful suggestions obtained from the many standard works on mechanics. An attempt has been made to give the specific reference to the original for material taken from engineering works or periodical literature. He wishes, moreover, to express his thanks to Dean C. H. Benjamin and Professor L. V. Ludy for their careful reading of the manuscript, to Professor W. K. Hatt for many of the problems used, and to Dean W. F. M. Goss, whose continued interest and advice have been a constant source of inspiration. It is hoped that the work may be an inspiration to students of engineering.

E. L. HANCOCK.

PURDUE UNIVERSITY,
November, 1908.

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APPLIED MECHANICS FOR ENGINEERS

CHAPTER I

DEFINITIONS

1. Introduction. — The study of the subject of mechanics of engineering involves a study of *matter*, *space*, and *time*. The subject as presented in this book consists of two parts; viz., *statics*, including the study of bodies under the action of systems of forces that are in equilibrium (balanced), and *dynamics*, including a study of the motion of bodies.

2. Force. — A body acted upon by the attraction or repulsion of another body is said to be subjected to an attractive or repulsive force, as the case may be. Forces are usually defined by the effects produced by them, as for example, we say, a force is something that produces motion or tends to produce motion, or changes or tends to change motion, or that changes the size or shape of a body. The study of relations between forces and the motions produced by them is usually designated as the study of *Statics* and *Dynamics*. Forces always occur in pairs; for example, a book held in the outstretched hand exerts a downward pressure on the hand, and the hand exerts an equal upward pressure on the book.

3. Unit of Force. — The unit of force used by engineers in this country and England is the pound avoirdupois. It is sometimes, however, necessary to use the *absolute unit of force*. This may be defined as follows: The absolute unit of force is that force which acting on a unit mass during unit time will produce in the mass, unit velocity. This absolute unit of force is called a *poundal*. In France, Germany, and other countries where the centimeter-gram-second system is used, the engineer's unit of force is the *kilogram*. The absolute unit of force, in such countries, is the force which acting upon a mass of one gram weight (at Paris) will produce a velocity of one centimeter per second, in a second. Such a unit is called a *dyne*.

4. Unit Weight. — The weight of a cubic foot of a substance will be called the *unit weight* of the substance and will be represented by γ. Below is given a table of such weights taken at the sea level. It will be seen that the unit weight of a substance divided by the unit weight of pure water gives its *specific gravity*. (See Table I on opposite page.)

5. Rigid Body. — In studying the state of motion or rest of a body due to the application of forces acting upon it, it is not necessary to consider the deformation of the body itself, due to the forces. When so considered it is customary to say that the body is a *rigid body*. Unless otherwise stated bodies will be considered as rigid bodies in this book.

6. Inertia. — The property of a body that causes it to continue in motion, if in motion, or remain at rest, if at rest, unless acted upon by some other force, is called *inertia*. This is Newton's First Law of Motion. (See Art. 76.)

TABLE I

UNIT WEIGHTS AND SPECIFIC GRAVITY OF SOME MATERIALS

(Kent's "Engineer's Pocket Book")

MATERIAL	SPECIFIC GRAVITY	UNIT WEIGHT
Brass	8.2 to 8.6	511 to 536
Brick		
Soft	1.6	100
Common	1.79	112
Hard	2.0	125
Pressed	2.16	135
Fire	2.24–2.4	140–150
Brickwork — mortar	1.6	100
Brickwork — cement	1.79	112
Concrete	1.92–2.24	120–140
Copper	8.85	552
Earth — loose	1.15–1.28	72–80
Earth — rammed	1.44–1.76	90–110
Granite	2.56–2.72	160–170
Gum	.92	57
Hickory	.77	48
Iron — cast	7.21	450
Iron — wrought	7.7	480
Lead	11.38	709.7
Limestone	2.72–3.2	170–200
Masonry — dressed	2.24–2.88	140–180
Nickel	8.8	548.7
Pine — white	.45	28
Pine — yellow	.61	38
Poplar	.48	30
Sandstone	2.24–2.4	140–150
Steel	7.85	490
White Oak	.77	48

7. Mass. — The mass of a body is the quantity of matter it contains. Mass differs from weight, in that the weight varies with the position on the surface of the earth and with the height above the surface, while the mass remains the same. The engineer's definition of mass, viz. that it is equal to the weight divided by the acceleration of gravity (see Art. 76), may be expressed $M = \frac{G}{g}$. Both G and g vary for different localities, but the quotient is constant; that is, the quantity of matter in a body is independent of its position with reference to the earth. The weight of a body may be determined by means of the spring balance. Such a balance is the only true measure of weight, since the equal-armed balance gives the same weight regardless of distance from the center of the earth. The equal-armed balance really measures mass.

8. Displacement. — By the displacement of a body is meant its change from one position to another. A displacement involves a movement in a definite direction. It may be represented by an arrow, the length of the arrow representing the distance moved and the direction of the arrow the direction of the motion. Thus, if a man walks due east one mile and then due north one mile, we might represent his displacement from the original position by an arrow drawn northeast of a length equal to $\sqrt{2}$ miles. Or, in Fig. 1, if P_2 represents a displacement of a body in the direction indicated and P_1 a subsequent displacement in the direction of P_1, then R represents a displacement equivalent to P_1 and P_2. It is seen that R may be determined by constructing a parallelogram on P_1 and P_2 as sides and drawing the diagonal. Quantities that may be represented by arrows are known as *vector quantities*, and the arrows themselves as *vectors*.

9. Representation of Forces. — Forces have a certain magnitude, act in a certain direction, and have a definite point of application. If a man, for example, attaches a rope to a log and pulls on the rope, his pull may be measured in pounds; it acts along the rope, and it has a point of application which is the same as the point of attachment of the rope to the log. It has been found convenient, for the purpose of analysis, to represent forces by arrows (vectors of Art. 8), the length of the arrow representing the magnitude of the force and the direction of the arrow giving the direction in which it acts. Thus, a 10-pound force, acting in a direction 30° with the horizontal, is represented by an arrow drawn in the same direction and having its point of application in the body and having a length representing 10 lb. (In this case, if 2 lb. represents 1 in., the length of the arrow is 5 in.) The line along which a force acts will be referred to as its *line of action.*

10. Concurrent Forces. — When two or more forces act upon the same point of a body, their lines of action are *concurrent*, and the forces are known as *concurrent forces.*

11. Resultant of Two Concurrent Forces. — If two forces having the same point of application act on a body, there is some single force that might be applied at the same point to produce the same effect. This single force is called the *resultant* of the two forces, and is found as follows: construct upon the arrows representing the forces a parallelogram and draw the diagonal from the point of application. This diagonal represents the resultant of the two forces in magnitude and direction

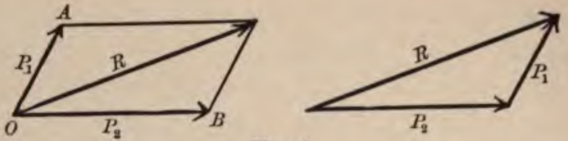


FIG. 1

(Art. 8). Thus, if P_1 and P_2 (Fig. 1) are the forces, then R is the resultant.

Algebraically $R = \sqrt{P_1^2 + P_2^2 + 2\,P_1P_2 \cos AOB}$.

12. Resolution of Force. — We have just seen how two concurrent forces may be replaced by a single force called their resultant. In a similar way a single force may be resolved into two forces. These forces are the sides of a parallelogram of which the single force is a diagonal. It is clear, then, that there are an infinite number of components into which a single resultant may be resolved. It is necessary, therefore, in speaking of the components of a force, to state specifically which are intended. It will be seen in problems that follow that the components most often used are at right angles to each other, and usually the *vertical* and *horizontal* components. In such a case the components are the *projections of the force on the vertical and horizontal lines.*

13. Force Triangle. — It follows directly from the parallelogram law of forces (Art. 11) that if we draw from any point a line parallel to and representing one of two concurrent forces, P_2 say, and from the extremity of this line another line parallel to P_1 and of the same length, then the remaining side of the triangle will be represented by R. This triangle is called the *force triangle.* In general, the resultant of two concurrent forces may be found by drawing lines parallel to the forces as above. The line necessary to complete the triangle is the resultant, and its arrow is always away from the point of application. The equal and opposite of this resultant would be a single force that would hold the two concurrent forces in equilibrium.

14. Force Polygon. — If more than two forces are concurrent, we may find their resultant by proceeding in a way similar to that outlined above. Thus, let the forces be P_1, P_2, P_3, P_4, etc. (Fig. 2), all passing through a point; from any point draw a line equal and parallel to P_1, from the extremity of the line draw another equal and parallel to P_2, from the extremity of this last line draw another equal and parallel to P_3, and proceed in the same way for the other forces. The figure produced will be a polygon whose sides are equal and parallel to the forces. The resultant will be given in magnitude, direction, and point of application by the line necessary to close the polygon. The arrow, representing the direction of the resultant, will always be away from the point of application. (See Fig. 2.) If the polygon be closed, the system of forces will be in equilibrium. The single force necessary to produce equilibrium will, in any case, be equal and opposite to R. The student should construct force polygons by taking the forces in different orders and checking the resultant in each case.

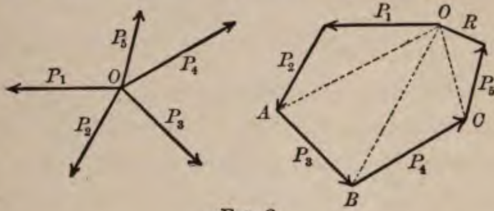


FIG. 2

By drawing the lines OA, OB, OC, etc., it is easy to see that OA represents the resultant of P_1 and P_2, that OB represents the resultant of OA and P_3, and so of P_1, P_2, and P_3, etc. That is, it is easy to see that the force polygon follows directly from the force triangle. By means of the force polygon it is easy to find graphically the resultant of any number of concurrent forces in a plane. The work, however, must be done accurately.

The student should show that the *force* polygon may be used for finding the resultant of concurring forces in space, by considering two forces at a time. The force polygon in this case is called a *twisted polygon.*

15. Transmissibility of Forces. — It is a matter of experience that the point of application of a force may be changed to any point along its line of action without changing the effect of the force upon the rigid body. This, of course, is on the assumption that all the force is transmitted to the body. The law may be stated as follows: *The point of application of a force may be transferred anywhere along its line of action without changing its effect upon the body upon which it acts.*

CHAPTER II

CONCURRENT FORCES

16. Concurrent Forces in a Plane. — It will often be convenient to consider forces as acting on a material point; this is equivalent to considering the body without weight and simply a point. If a material point (O) (Fig. 3) be acted upon by a number of forces in a plane, P_1, P_2, P_3, P_4, etc., each one making angles α_1, α_2, α_3, α_4, etc., respectively, with the positive x-axis, it is desirable to find the resultant of all of them in magnitude and direction; that is, the single ideal force that could produce the same effect as the system of forces.

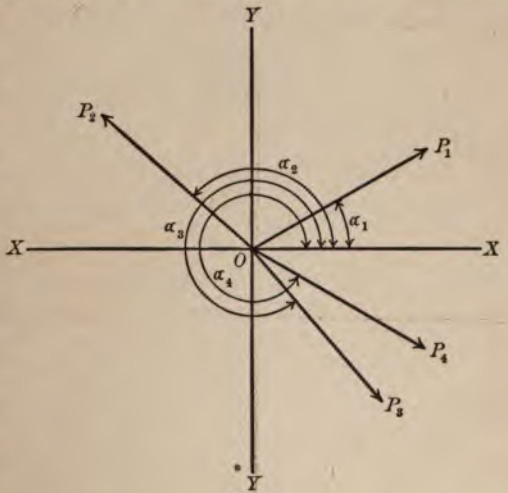


FIG. 3

Each force P may be resolved into components along the x- and y-axes, giving $P \cos \alpha$ along the x-axis, and P

$\sin \alpha$ along the y-axis. The sum of these components along the x-axis may be expressed,

$$\Sigma x = P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + P_3 \cos \alpha_3 + \text{etc.},$$

the proper algebraic sign being given $\cos \alpha$ in each case. In a similar way the sum of the components along the y-axis may be written,

$$\Sigma y = P_1 \sin \alpha_1 + P_2 \sin \alpha_2 + P_3 \sin \alpha_3 + \text{etc.}$$

These forces, Σx and Σy, may now replace the original system as shown in Fig. 4. And these may be combined into a single resultant which is the diagonal of the rectangle of which the two forces are sides (Art. 11). This gives the resultant in magnitude and direction, and this resultant force is the single force which if allowed to act upon the material point would produce the same effect as the system of forces. It should be remembered in all that follows that this resultant force has no real existence; it

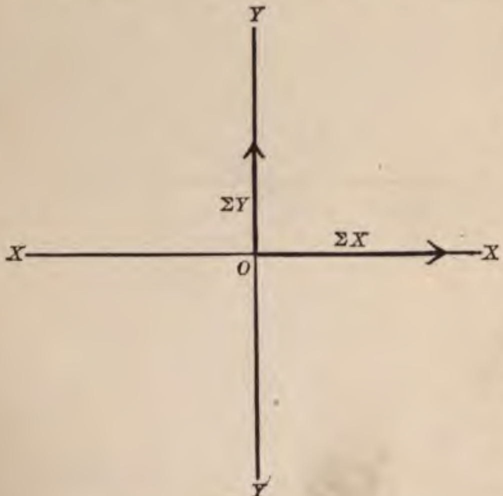


FIG. 4

is used to simplify the solution of problems. Analytically the resultant may be expressed,

$$R = \sqrt{(\Sigma x)^2 + (\Sigma y)^2},$$

and its direction α as such an angle that $\tan \alpha = \frac{\Sigma y}{\Sigma x}$. (See Fig. 5.) If the material point be at rest or moving uniformly, this resultant force must be equal to zero; that is,

$$\sqrt{(\Sigma x)^2 + (\Sigma y)^2} = 0.$$

This means that $(\Sigma x)^2 + (\Sigma y)^2 = 0$, that is, that the sum of two squares must be zero; but this can happen only when each one, separately, is zero (since neither can be negative being squared). We therefore have as the necessary and sufficient conditions for the equilibrium of a material point, acted upon by a system of concurring forces in a plane,

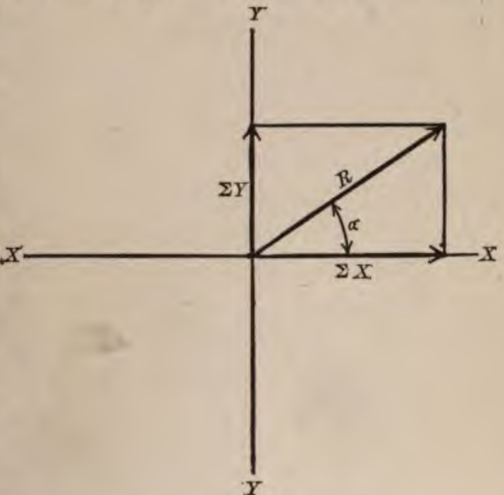


FIG. 5

$$\boldsymbol{R = 0 \text{ or } \Sigma x = 0, \text{ and } \Sigma y = 0.}$$

When R is not zero, the system of forces causes accelerated motion in the direction of R; when $R = 0$, the material point remains at rest, if at rest, or continues in motion with uniform velocity, if in motion. In this case the system of forces is said to be *balanced*.

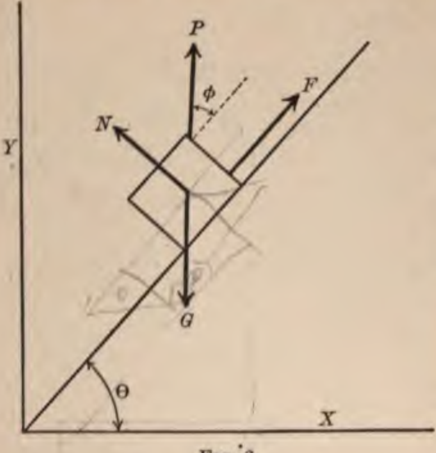


FIG. 6

As an illustration of the foregoing, consider the case of a body of weight G situated on an inclined plane, making an angle θ with the horizontal. (See Fig. 6.) There is a certain force P making an angle ϕ with the plane, whose component along the plane acts upwards, and also a force of friction F upwards. The other forces acting on the body are G, the force of gravity acting vertically, and N, the normal pressure of the plane. Taking the x-axis along the plane positive upward and the y-axis perpendicular to it positive upward, we get,

$$\Sigma x = P\cos\phi + F - G\sin\theta,$$

and

$$\Sigma y = N + P\sin\phi - G\cos\theta.$$

For equilibrium

$$P\cos\phi + F - G\sin\theta = 0,$$

$$N + P\sin\phi - G\cos\theta = 0.$$

Therefore,

$$N = G\cos\theta - P\sin\phi,$$

$$P = \frac{G\sin\theta - F}{\cos\phi}.$$

This last equation gives the magnitude of P required to preserve equilibrium, supposing that the force of friction, θ, and ϕ are known.

Problem 1. An angle iron whose weight is 20 lb. and angle a right angle, rests upon a circular shaft, radius 2 in. Find the normal pressure at A and B (Fig. 7).

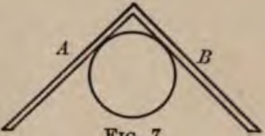


FIG. 7

Problem 2. Given three concurring forces, 100 lb., 50 lb., and 200 lb., whose directions referred to the x-axis are 0°, 60°, 180°, respectively; find the resultant in magnitude and direction.

Problem 3. A body (Fig. 8) whose weight is G is drawn up the inclined plane with uniform velocity due to the action of the forces P and P'. Find the force of friction and the normal pressure, if $P = 100$ lb., $P' = 100$ lb., $G = 160$ lb. P acts parallel to the plane and P' acts horizontally.

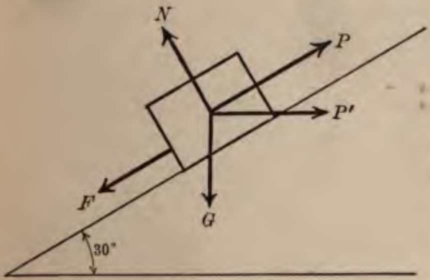


FIG. 8

Problem 4. A wheel is about to roll over an obstruction. The diameter of the wheel (Fig. 9) is 3' and its weight 800 lb. Find the force P necessary to start the wheel over the obstruction.

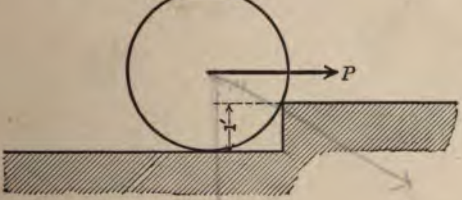


FIG. 9

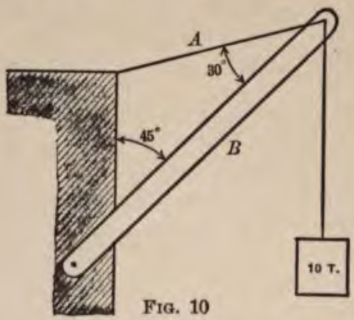


FIG. 10

Problem 5. A weight of 10 tons is supported as shown in Fig. 10. Find the force acting in the tie A and the member B.

17. Concurrent Forces in Space. — If the material point (O) be acted upon by a system of concurrent forces not in a plane P_1, P_2, P_3, P_4, etc., whose direction angles are $\alpha_1\beta_1\gamma_1$, $\alpha_2\beta_2\gamma_2$, $\alpha_3\beta_3\gamma_3$, $\alpha_4\beta_4\gamma_4$, etc., respectively, one of which is shown in Fig. 11, the resultant force may be found in magnitude and direction by an analysis similar to that used in the preceding case. The sum of the components of all the forces along the x-axis is

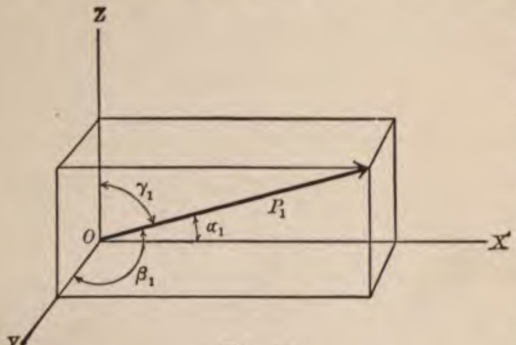


FIG. 11

$$\Sigma x = P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + P_3 \cos \alpha_3 + P_4 \cos \alpha_4 + \text{etc.}$$

Similarly, the sum of the components along the y-axis,

$$\Sigma y = P_1 \cos \beta_1 + P_2 \cos \beta_2 + P_3 \cos \beta_3 + P_4 \cos \beta_4 + \text{etc.},$$

and the sum of the components along the z-axis,

$$\Sigma z = P_1 \cos \gamma_1 + P_2 \cos \gamma_2 + P_3 \cos \gamma_3 + P_4 \cos \gamma_4 + \text{etc.}$$

The original system of forces may now be replaced by a system of three rectangular forces Σx, Σy, and Σz (Fig. 12). Finally, this system may be replaced by a resultant which is the diagonal of a parallelopiped constructed with Σx, Σy, and Σz as edges. In magnitude this resultant may be expressed

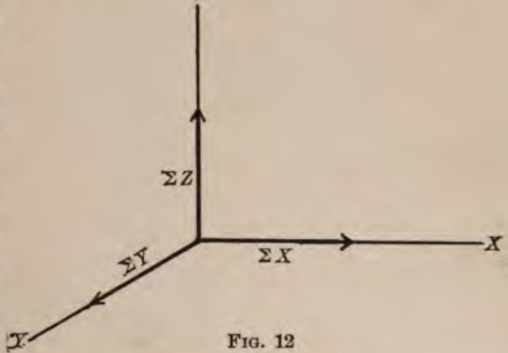


FIG. 12

$$R = \sqrt{(\Sigma x)^2 + (\Sigma y)^2 + (\Sigma z)^2}, \qquad \text{(See Fig. 13)}$$

and its direction given by the angles α, β, and γ. These angles are given by the equations

$$\cos \alpha = \frac{\Sigma x}{R}, \ \cos \beta = \frac{\Sigma y}{R}, \ \cos \gamma = \frac{\Sigma z}{R}.$$

For equilibrium R must be 0; that is,

$$(\Sigma x)^2 + (\Sigma y)^2 + (\Sigma z)^2 = 0,$$

and therefore,

$$\mathbf{\Sigma x = 0,\ \Sigma y = 0,\ \Sigma z = 0.}$$

This gives three equations of condition from which three unknown quantities may be determined. In the preceding case of Art. 16 there were only two equations of condition $\Sigma x = 0$ and $\Sigma y = 0$; consequently, only two unknown quantities could be determined.

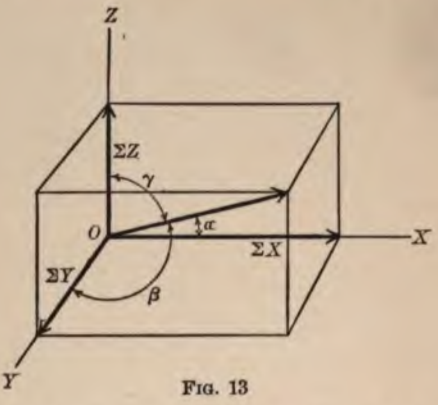


FIG. 13

Problem 6. Three men (Fig. 14) are each pulling with a force P at the points a, b, and c, respectively. What weight Q can they raise with uniform motion if each man pulls 100 lb.? Each force makes an angle of 60° with the horizontal.

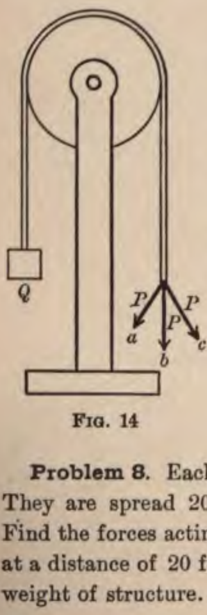


FIG. 14

Problem 7. Three concurring forces act upon a rigid body. Find the resultant in magnitude and direction. The forces are defined as follows:

$P_1 = 75$ lb.; $\alpha_1 = 63° \, 27'$; $\beta_1 = 48° \, 36'$; $\gamma_1 = ?$

$P_2 = 80$ lb.; $\alpha_2 = 153° \, 44'$; $\beta_2 = 67° \, 13'$; $\gamma_2 = ?$

$P_3 = 95$ lb.; $\alpha_3 = 76° \, 14'$; $\beta_3 = 147° \, 2'$; $\gamma_3 = ?$

HINT. γ_1, γ_2, and γ_3 may be found from either of the following relations:

$$\cos(\alpha + \beta)\cos(\alpha - \beta) + \cos^2\gamma = 0,$$
$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

Problem 8. Each leg of a pair of shears (Fig. 15) is 50 ft. long. They are spread 20 ft. at the foot. The back stay is 75 ft. long. Find the forces acting on each member when lifting a load of 20 tons at a distance of 20 ft. from the foot of the shear legs, neglecting the weight of structure.

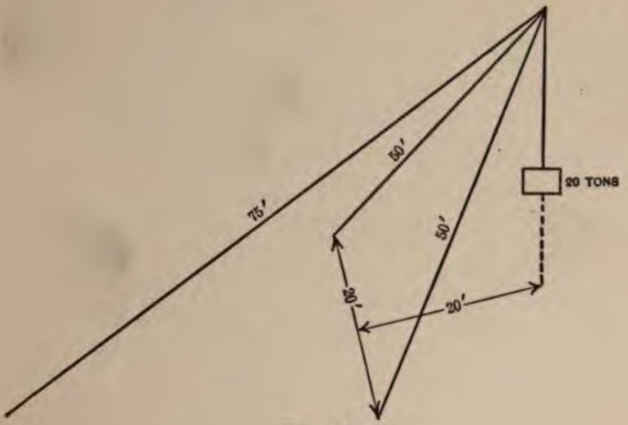


FIG. 15

18. Moment of a Force. — *The moment of a force with respect to any point in its plane may be defined as the product of the force and a perpendicular let fall from the point on the line of action of the force.* Let P (Fig. 16) be the force and O the point and a the perpendicular distance of the force from the point; then Pa is the moment of the force with respect to the point O. This moment is measured in terms of the units of both *force* and *length*, viz. foot-pounds or inch-pounds, and is read foot-pounds moment or inch-pounds moment to distinguish it from foot-pounds work or inch-pounds work.

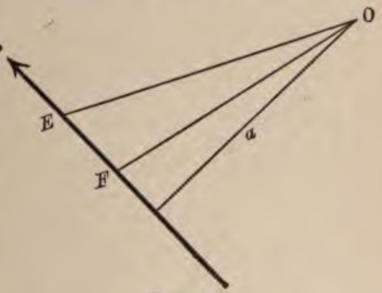


FIG. 16

For convenience the algebraic sign of the moment is said to be *positive* when the moment tends to turn the body in a direction *counter-clockwise*, and *negative* when it tends to turn the body in the *clockwise* direction.

The moment may be represented geometrically as follows: let EF represent the magnitude of P, drawn to the desired scale, and draw EO and FO. The area of the triangle $OEF = \frac{1}{2}EFa$, or $EFa = 2\triangle OEF$; that is, *the moment of the force with respect to a point is geometrically represented by twice the area of the triangle, whose base is the line representing the magnitude of the force and whose vertex is the given point.*

19. Varignon's Theorem of Moments. — *The moment of the resultant of two concurring forces with respect to any point in their plane is equal to the algebraic sum of the moments of the two forces with respect to the same point.*

The given forces P and P_1 may be represented by CP and CP_1 and their resultant by CR. Take O as the point with respect to which moments are to be taken, and construct the dotted lines as shown in Fig. 17. The moment of CP with respect to O is twice the area of the triangle COP (Art. 18), $= 2$ area of the triangle COB, since the tri-

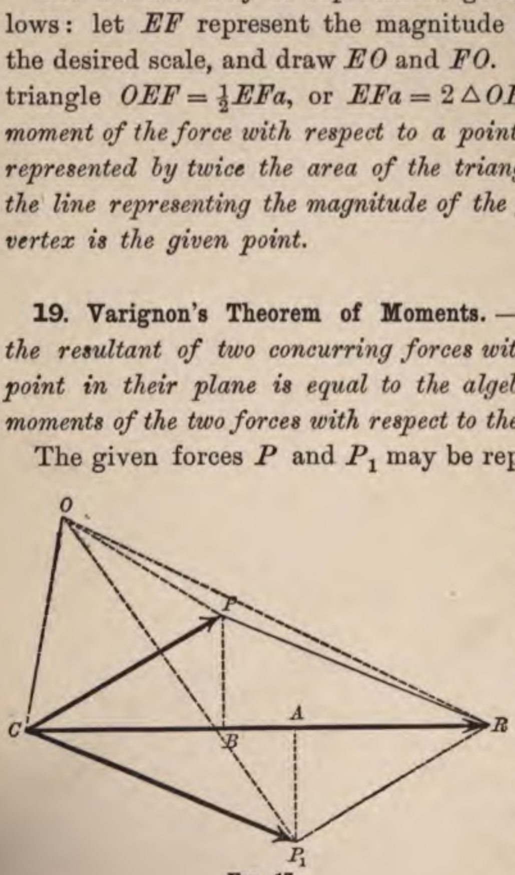


FIG. 17

angles have the same base and the same altitude. That is, the moment of CP with respect to O is the same as the moment of CB with respect to $O = CBa$, where a is the perpendicular let fall from O on CR. In a similar way, it is seen that the moment of CP_1 with respect to O is equal to the moment of CA with respect to O; that is, to $CA \cdot a$. Therefore, the sum of the moments of P and P_1 with respect to O equals $(CB + CA) \cdot a$. But $(CB + CA)a = (CA + AR) \cdot a = R \cdot A$, since $CB = AR$ (equal triangles CPB and AP_1R). When the point is taken between the P and P_1, the moment of the resultant equals the difference of the moments of P and P_1. Let the student show that this is true.

Cor. 1. If there are any number of concurring forces in a plane, it may be shown that Varignon's theorem holds by considering the resultant of two of them with the third, and so on. The more general theorem may then be stated as follows: *The moment of the resultant of any number of concurring forces in a plane with respect to any point in that plane is equal to the algebraic sum of the moments of the forces with respect to the same point.*

Cor. 2. If the point be taken in the line of action of R, then $a = 0$, and therefore the sum of the positive moments equals the sum of the negative moments.

The moment of a force with respect to a line at right angles to the line of action of the force is the product of the force and the shortest distance between the two lines.

The moment of a force with respect to a line not at right angles to the line of action of the force is the same as the moment of the component of the force in a plane perpendicular to the line.

CHAPTER III

PARALLEL FORCES

20. The Resultant of Two Parallel Forces. — In considering two parallel forces in a plane three cases arise: (*a*) when the forces are in the same direction; (*b*) when they are *unequal* and in opposite directions; (*c*) when they are *equal* and in opposite directions, but having different lines of action.

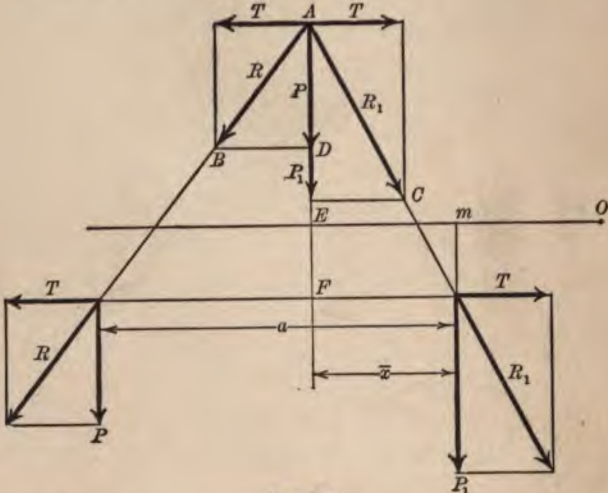


FIG. 18

CASE (*a*). When the forces are in *same direction.* The two forces are P and P_1, and the distance between their lines of action is a. For the sake of analysis put in two equal and opposite forces T as shown in Fig. 18. These forces will have no effect as far as the state of motion of the body is concerned. The resultant of T and P is R, and that of T and P_1 is R_1. Transfer R and R_1 to the point of intersection of their lines of action A. Here resolve them into components parallel to their original components; the two forces T nullify each other, and there are left the two forces P and P_1 acting along the same line AE. The resultant of P and P_1, then, is equal to $P + P_1$ and acts in the *same* direction as the forces.

To determine the position of R with reference to the forces we have from similar triangles

$$\frac{\bar{x}}{AF} = \frac{T}{P_1} \text{ and } \frac{a - \bar{x}}{AF} = \frac{T}{P},$$

from which $$\frac{P_1}{P} = \frac{a - \bar{x}}{x}, \text{ or } \bar{x} = \frac{Pa}{R}.$$

That is, the resultant of two parallel forces in the same direction divides the distance between them in the inverse ratio of the forces.

Cor. For any point O in the same plane, it is easy to show that the moment of the resultant is equal to the algebraic sum of the moments of the two forces with respect to this same point. Draw a line through O parallel to T and let m be the distance from O to the line of action of P_1. If now Rm be added to both sides of the equation

$$R\bar{x} = Pa,$$

we shall have

$$R(m + \bar{x}) = P(a + m) + P_1 m,$$

and this is the relation we were to find.

When O is a point on the line of action of R, the moment of $R = 0$, and we have the moment of P equal to the moment of P_1. This is often a convenient relation to use in the solution of problems. Following out the above reasoning, let the student show *that the moment of the resultant of any number of parallel forces in a plane with respect to a point in the plane is equal to the algebraic sum of the moments of the forces with respect to that same point.*

CASE (*b*). When the forces are *unequal* and *opposite* in direction. In this case the analysis is exactly similar to Case (*a*) and leads to exactly the same conclusions. It is left as an exercise to be worked out by the student.

CASE (*c*). When the forces are *equal* and *opposite*, but not acting along the same line, they form a *couple*. These will be treated later. (See Art. 30.)

Problem 9. Two parallel forces, one of 20 lb. and one of 100 lb., have lines of action 24 in. apart. Find the resultant in magnitude, direction, and point of application:

(1) When they are in the same direction.

(2) When they are in opposite directions.

Problem 10. A horizontal beam of length l is supported at its ends by two piers and loaded with a single load P at a distance of $\frac{l}{5}$ from one end. Find the pressure of the piers against the beam.

Problem 11. The locomotive shown in Fig. 19 is run upon a turntable whose length is 100 ft. Find the position of the engine so that the table will balance.

21. System of Parallel Forces in Space. — If the forces are all parallel, it is evident that the resultant is equal in magnitude to the algebraic sum of the forces, and that its line of action is parallel to the forces. It remains, then, to determine the point of application of this resultant.

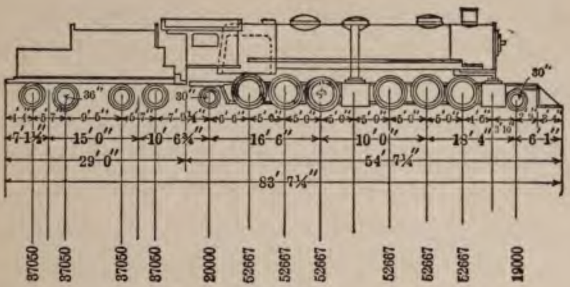


FIG. 19

Suppose the forces represented by P_1, P_2, P_3, P_4, etc., and let their points of application be $x_1y_1z_1$, $x_2y_2z_2$, $x_3y_3z_3$, $x_4y_4z_4$, etc. (Fig. 20). (In order to avoid a complicated figure only two forces are shown.) The two forces

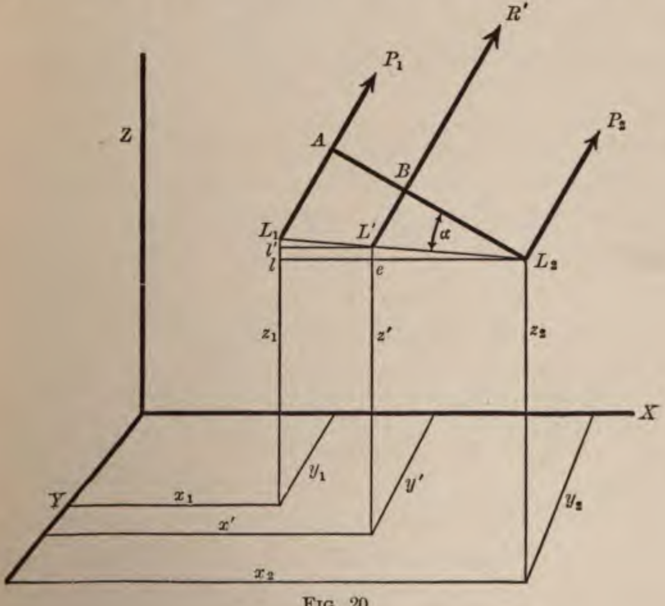


FIG. 20

P_1 and P_2 lie in a plane and have a resultant $R' = P_1 + P_2$ whose point of application is at a distance z' from the xy-plane and on a line joining L_1 and L_2 at L'. Draw L_2A perpendicular to L_1A. Then from Art. 20, $R'L_2B = P_1L_2A$, which multiplied by sec α gives $R'L_2L' = P_1L_2L_1$,

or
$$\frac{L_2L'}{L_2L_1} = \frac{P_1}{P_1 + P_2}.$$

Now in the plane of z_1 and z_2 draw L_2l and $L'l'$ perpendicular to z_1, and we have

$$\frac{L_2L'}{L_2L_1} = \frac{L'e}{L_1l} = \frac{z' - z_2}{z_1 - z_2},$$

so that
$$z' - z_2 = \frac{P}{P_1 + P_2}(z_1 - z_2),$$

$$z' = \frac{P_1z_1 + P_2z_2}{P_1 + P_2}.$$

Consider now R' with P_3; these forces lie in a plane. Let their resultant be R'' and its point of application x'', y'', z''. Following out the above reasoning for this case, the resultant is seen to be $R'' = P_1 + P_2 + P_3$ and

$$z'' = \frac{R'z' + P_3z_3}{R' + P_3} = \frac{P_1z_1 + P_2z_2 + P_3z_3}{P_1 + P_2 + P_3}.$$

Extending this process so as to include all of the forces P_4, P_5, etc., and calling the final resultant R and its point of application $\bar{x}$, $\bar{y}$, $\bar{z}$, we have

$$R = P_1 + P_2 + P_3 + P_4 + \text{etc.}$$

and
$$\bar{z} = \frac{P_1z_1 + P_2z_2 + P_3z_3 + P_4z_4 + \text{etc.}}{P_1 + P_2 + P_3 + P_4 + \text{etc.}} = \frac{\Sigma Pz}{\Sigma P},$$

and by a reasoning similar to the above

$$\bar{y} = \frac{P_1y_1 + P_2y_2 + P_3y_3 + P_4y_4 + \text{etc.}}{P_1 + P_2 + P_3 + P_4 + \text{etc.}} = \frac{\Sigma Py}{\Sigma P}.$$

$$\bar{x} = \frac{P_1x_1 + P_2x_2 + P_3x_3 + P_4x_4 + \text{etc.}}{P_1 + P_2 + P_3 + P_4 + \text{etc.}} = \frac{\Sigma Px}{\Sigma P}.$$

This point of application of the resultant is called the *center of the system of parallel forces.*

As an illustration of the above, suppose $P_1 = 50$ lb., $P_2 = 100$ lb., $P_3 = 300$ lb., $P_4 = 10$ lb., and $P_5 = -400$ lb., and their points of application respectively $2, 1, -5$; $-1, -2, 4$; $2, 1, -2$; $-2, 1, 1$; $1, 1, 1$. The resultant in this case equals 50 lb. + 100 lb. + 300 lb. + 10 lb. − 400 lb. = 60 lb. and its point of application

$$\bar{x} = \frac{50(2) + 100(-1) + 300(2) + 10(-2) - 400(1)}{60} = 3,$$

$$\bar{y} = \frac{50(1) + 100(-2) + 300(1) + 10(1) - 400(1)}{60} = -4,$$

$$\bar{z} = \frac{50(-5) + 100(4) + 300(-2) + 10(1) - 400(1)}{60} = -14.$$

As another illustration consider the problem of finding the center of the system of parallel forces P_1, P_2, P_3, in Fig. 21. The figure represents a Z-iron of the same cross section throughout, and P_1, P_2, and P_3 are therefore the weights of the individual parts (considering the Z-iron as divided into three parts — two legs and the connecting vertical portion). If the weight of a cubic inch of iron = .26 lb., $P_1 = .78$ lb., $P_2 = 2.08$ lb., $P_3 = 1.04$ lb., and therefore $R = 3.9$ lb. The points of application of P_1, P_2, and P_3

are $(-\frac{1}{2}, -\frac{1}{2}, 9\frac{1}{2})$, $(\frac{1}{2}, -\frac{1}{2}, 5)$, and $(2, -\frac{1}{2}, \frac{1}{2})$, respectively, so that

$$\bar{x} = \frac{.78(-\frac{1}{2}) + 2.08(\frac{1}{2}) + 1.04(2)}{3.9} = .70 \text{ in.},$$

$$\bar{y} = \frac{.78(-\frac{1}{2}) + 2.08(-\frac{1}{2}) + 1.04(-\frac{1}{2})}{3.9} = -.50 \text{ in.},$$

$$\bar{z} = \frac{.78(\frac{19}{2}) + 2.08(5) + 1.04(\frac{1}{2})}{3.9} = 4.7 \text{ in.}$$

This point $\bar{x}$, $\bar{y}$, $\bar{z}$ is, in this case, the center of gravity of the Z-iron.

Problem 12. Parallel forces of P_1, P_2, P_3, and P_4 act at the corners of a rectangle 3 ft. by 2 ft. and perpendicular to its plane. Find the point of application of the resultant, if $P_1 = 10$ lb., $P_2 = 50$ lb., $P_3 = 100$ lb., $P_4 = 200$ lb., P_1 and P_2 being 2 ft. apart, and P_3 on same side as P_2.

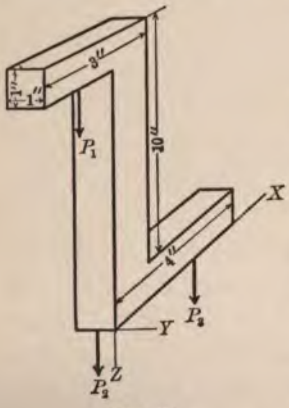


FIG. 21

Problem 13. Eight parallel forces act at the corners of a one-inch cube, making an angle of 45° with one of its faces. Find the point of application of the resultant force, if $P_1 = 30$ lb., $P_2 = 50$ lb., $P_3 = 10$ lb., $P_4 = 20$ lb., $P_5 = 100$ lb., $P_6 = 5$ lb., $P_7 = 10$ lb., $P_8 = 40$ lb.

The student should prove that the moment of the resultant of any system of parallel forces in space with respect to any line in space, equals the sum of the moments of the forces with respect to this same line. The solution of Problems 12 and 13 is made much shorter by using this principle.

CHAPTER IV

CENTER OF GRAVITY

22. Definition of the Center of Gravity. — The center of gravity of a body may be defined as the point of application of the resultant attraction of the earth for that body, and the center of gravity of several bodies considered together, as the point of application of the resultant attraction of the earth for the bodies. The expressions for $\bar{x}$, $\bar{y}$, and $\bar{z}$, Art. 21, may be used for locating the center of gravity, in the latter case, without change, P_1, P_2, P_3, etc., representing the weights of the individual bodies. In such cases *the center of the system of parallel forces is the center of gravity of the body.* The attention of the student is called to the fact that the forces acting upon the particles of a body, due to the attraction of the earth, are not parallel, but meet in the center of the earth. For all practical purposes, however, they are considered parallel.

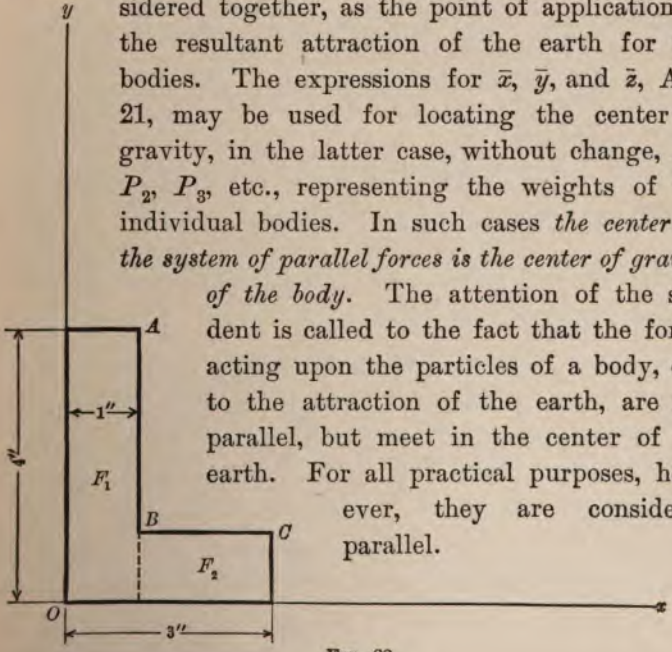


FIG. 22

If the unit weight times the volume be substituted for weight, that is, if we write instead of P_1, $\gamma_1 V_1$ and P_2, $\gamma_2 V_2$, etc., then $\bar{x}$, $\bar{y}$, $\bar{z}$ become

$$\bar{x}=\frac{\gamma_1 V_1 x_1+\gamma_2 V_2 x_2+\gamma_3 V_3 x_3+\text{etc.}}{\gamma_1 V_1+\gamma_2 V_2+\gamma_3 V_3+\text{etc.}}=\frac{\Sigma\gamma V x}{\Sigma\gamma V},$$

$$\bar{y}=\frac{\Sigma\gamma V y}{\Sigma\gamma V},\quad \bar{z}=\frac{\Sigma\gamma V z}{\Sigma\gamma V}.$$

And if the bodies are all of the same material and so have the same heaviness, γ is constant and may be taken outside the summation sign, where it cancels out. This gives values for $\bar{x}$, $\bar{y}$, and $\bar{z}$,

$$\bar{x}=\frac{\Sigma V x}{\Sigma V},\ \bar{y}=\frac{\Sigma V y}{\Sigma V},\ \bar{z}=\frac{\Sigma V z}{\Sigma V},$$

formulæ exactly similar to those of Art. 21, where the P's are replaced by V's.

If the bodies are thin plates of the same material, of constant thickness b, we may write for V_1, V_2, V_3, etc., bF_1, bF_2, bF_3, etc., where the F's represent the areas of the faces of the plates. Making this substitution for the V's, $\bar{x}$, $\bar{y}$, $\bar{z}$ may be written

$$\bar{x}=\frac{bF_1x_1+bF_2x_2+bF_3x_3+\text{etc.}}{bF_1+bF_2+bF_3+\text{etc.}}=\frac{\Sigma F x}{\Sigma F},$$

$$\bar{y}=\frac{\Sigma F y}{\Sigma F},\quad z=\frac{\Sigma F z}{\Sigma F};$$

the b being a constant factor, cancels out. These formulæ are applicable to finding the center of gravity of areas, and are much used by engineers for finding the center of gravity of sections of angles, channels, T-sections, Z-sec-

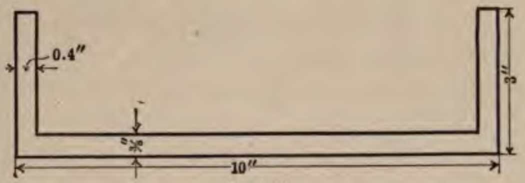


Fig. 23

tions, etc. As an illustration let it be required to find the center of gravity of the angle section shown in Fig. 22. It is convenient to select the x- and y-axes as shown and to divide the area up into the two indicated areas F_1 and F_2. We then have

$$\bar{x} = \frac{F_1x_1 + F_2x_2}{F_1 + F_2} \text{ and } \bar{y} = \frac{F_1y_1 + F_2y_2}{F_1 + F_2},$$

x_1y_1 being the center of gravity of F_1, and x_2y_2 the center of gravity of F_2. It is left to the student to make the numerical substitution and to calculate the values for $\bar{x}$ and $\bar{y}$.

Second Method. The same results for $\bar{x}$ and $\bar{y}$ might be obtained from the expressions

$$\bar{x} = \frac{F_1x_1 - F_2x_2}{F_1 - F_2}, \quad \bar{y} = \frac{F_1y_1 - F_2y_2}{F_1 - F_2},$$

where now F_1 is the area formed by completing the rectangle whose sides are 4 in. and 3 in. and x_1y_1 the center of gravity of this rectangle referred to the coördinate axes, and F_2 the area of the rectangle whose sides are 3 in. and 2 in. and x_2y_2 the coördinates of its center of gravity referred to the same axes. Let the student find the center of gravity of the angle section by this method and compare the results with those obtained by the previous method.

Problem 14. Find the center of gravity of the channel section shown in Fig. 23.

Problem 15. Find the center of gravity of the T-section shown in Fig. 24.

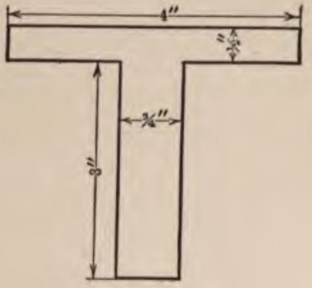


FIG. 24

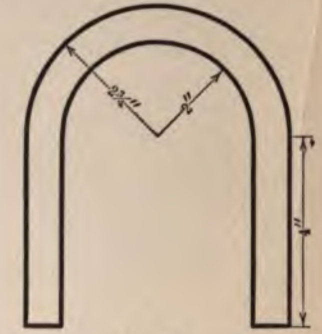


FIG. 25

Problem 16. Find the center of gravity of the U-section shown in Fig. 25. Given the fact that the center of gravity of a semicircular area is $\frac{4r}{3\pi}$ from the diameter. (See Prob. 22.)

Problem 17. Find the position of the center of gravity of a trapezoidal area, the lengths of whose parallel sides are a_1 and a_2, respectively, and the distance between them h. (See Fig. 26.)

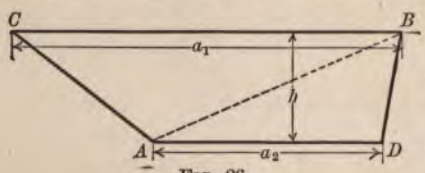


FIG. 26

HINT. Draw the diagonal AB and call the triangle ACB, F_1, and the triangle ABD, F_2. Given, the center of gravity of a triangle is ⅓ the distance from the base to the vertex. (See Prob. 21.) Select AD as the x-axis, then

$$\bar{y} = \frac{F_1 y_1 + F_2 y_2}{F_1 + F_2},$$

where $y_1 = \frac{2}{3}h$ and $y_2 = \frac{1}{3}h$. The center of gravity is seen to lie on a line joining the middle points of the parallel sides.

Problem 18. A cylindrical piece of cast iron whose height is 6 in. and the radius of whose base is 2 in., has a cylindrical hole of 1 in. radius drilled in one end, the axis of which coincides with the axis of the cylinder. The hole was originally 3 in. deep, but has been filled with lead until it is only 1 in. deep. Find the center of gravity of the body, the unit weight of lead being 710 and of cast iron 450. (Fig. 27.)

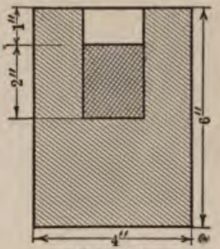


FIG. 27

Problem 19. Find the center of gravity of a portion of a reinforced concrete beam. (See Fig. 28.) The beam is reinforced with three half-inch steel rods, centers 1 in. from the bottom of the beam and 1 in. from the sides. The center of the middle rod is 4 in. from the sides.

(γ for steel = 490 lb. per cubic foot;
γ for concrete = 125 lb. per cubic foot.)

NOTE. It is seen that the thickness cancels out of the expression for the center of gravity, and might, therefore, have been neglected.

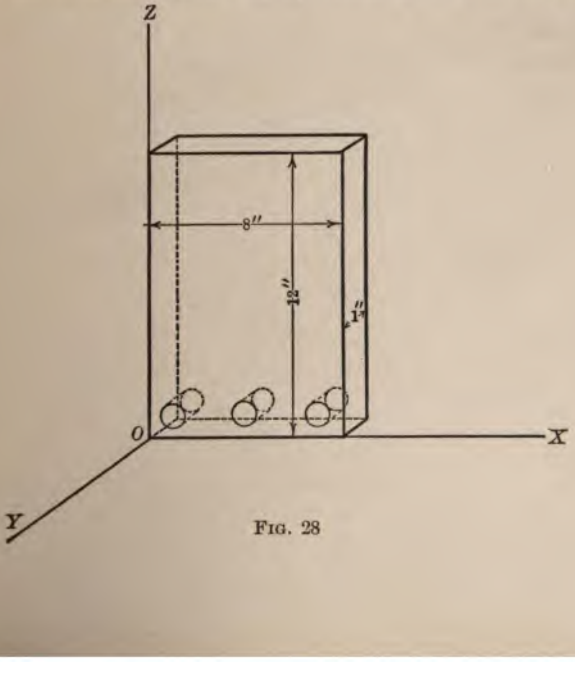


FIG. 28

23. Center of Gravity determined by Symmetry. — In some areas and solids it is often possible to determine the center of gravity from considerations of the symmetry of the figure; for example, the center of gravity of a parallelogram is at its geometrical center. This is also true of the circle, square, cylinder, sphere, etc. Whenever any axis is an axis of symmetry, that is, an axis such that for every element of area or volume on one side there is an equal area or volume on the other side, symmetrically placed, the center of gravity must be on that axis. This was found to be true in the case of the channel section, the T-section, and the U-section. In each of these cases the vertical line through the center of gravity is an axis of symmetry. The student will be able to note many more such cases, and by a little thought will often be able to find either $\bar{x}$, $\bar{y}$, or $\bar{z}$ from observation.

24. Center of Gravity determined by Aid of the Calculus. — The expressions for $\bar{x}$, $\bar{y}$, and $\bar{z}$ used to locate the center of gravity in Art. 22 may be put in the form of the quotient of two integrals, and these expressions may be integrated when there is no discontinuity in the expressions between the limits taken. With this understanding we may write

$$\bar{x} = \frac{\Sigma Px}{\Sigma P} = \frac{\int dP(x)}{\int dP},$$

$$\bar{y} = \frac{\Sigma Py}{\Sigma P} = \frac{\int dP(y)}{\int dP},$$

$$\bar{z} = \frac{\Sigma Pz}{\Sigma P} = \frac{\int dP(z)}{\int dP}.$$

As an illustration, suppose it is desired to obtain the center of gravity of a right circular cone of altitude h and radius of base r. Take the x-axis as the axis of the cone with the vertex at the origin. (See Fig. 29.) It is evident that $\bar{y} = 0$ and $\bar{z} = 0$, so that it is only necessary to find $\bar{x}$. The volume of any dv cut from the cone by two parallel planes, perpendicular to x and separated by a distance dx, is $\pi y^2 dx$, and the weight of this dv is $\gamma\pi y^2 dx = dP$. Therefore

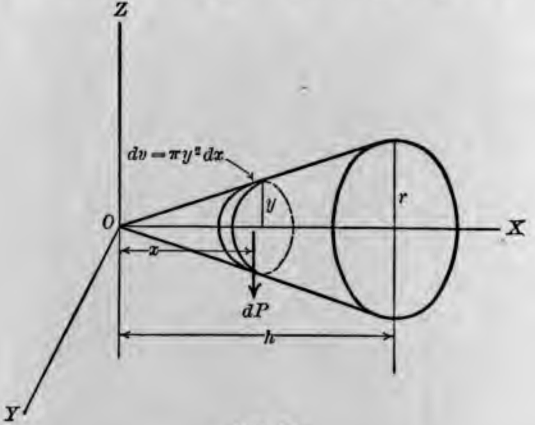


FIG. 29

$$\bar{x} = \frac{\int x\,dP}{\int dP} = \frac{\int_0^h x\gamma\pi y^2 dx}{\int_0^h \gamma\pi y^2 dx}.$$

But from similar triangles $y : x :: r : h$ or $y = \frac{r}{h}x$. This gives

$$\bar{x} = \frac{\gamma\pi\frac{r^2}{h^2}\int_0^h x^3 dx}{\gamma\pi\frac{r^2}{h^2}\int_0^h x^2 dx} = \frac{\left.\frac{x^4}{4}\right]_0^h}{\left.\frac{x^3}{3}\right]_0^h} = \frac{3}{4}h.$$

The expressions for $\bar{x}$, $\bar{y}$, and $\bar{z}$, involving dP, may be changed to similar ones involving dv, and these become for homogeneous bodies, since $dP = \gamma dv$,

$$\bar{x} = \frac{\int x dv}{\int dv}, \quad \bar{y} = \frac{\int y dv}{\int dv}, \quad \bar{z} = \frac{\int z dv}{\int dv},$$

and for thin plates of constant thickness b the dv may be replaced by bdF, giving values of $\bar{x}$, $\bar{y}$, and $\bar{z}$ for area,

$$\bar{x} = \frac{\int x dF}{\int dF}, \quad \bar{y} = \frac{\int y dF}{\int dF}, \quad \bar{z} = \frac{\int z dF}{\int dF}.$$

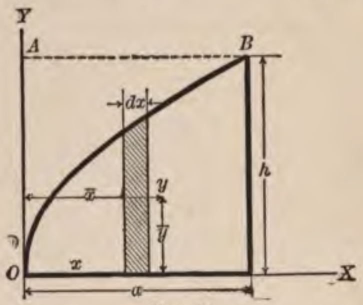


FIG. 30

The center of gravity of thin homogeneous wires of constant cross section may be found by replacing the dv in the above formulæ by ads, where a is the constant area of cross section and ds is a distance along the curve. The formulæ then become

$$\bar{x} = \frac{\int x ds}{\int ds}, \quad \bar{y} = \frac{\int y ds}{\int ds}, \quad \bar{z} = \frac{\int z ds}{\int ds}.$$

Problem 20. Find the center of gravity of a parabolic area shown in Fig. 30, the equation of the parabola being $y^2 = 2px$.

$$\bar{x} = \frac{\int x\,dF}{\int dF}, \quad \bar{y} = \frac{\int y\,dF}{\int dF}.$$

Here $dF = ydx$, so that

$$\bar{x} = \frac{\int_0^a xydx}{\int_0^a ydx}, \quad = \frac{\sqrt{2p}\int_0^a x^{\frac{3}{2}}dx}{\sqrt{2p}\int_0^a x^{\frac{1}{2}}dx} = \frac{\frac{2}{5}x^{\frac{5}{2}}\Big]_0^a}{\frac{2}{3}x^{\frac{3}{2}}\Big]_0^a} = \frac{3}{5}a.$$

It is left as a problem for the student to show that $\bar{y} = \frac{3}{8}h$.

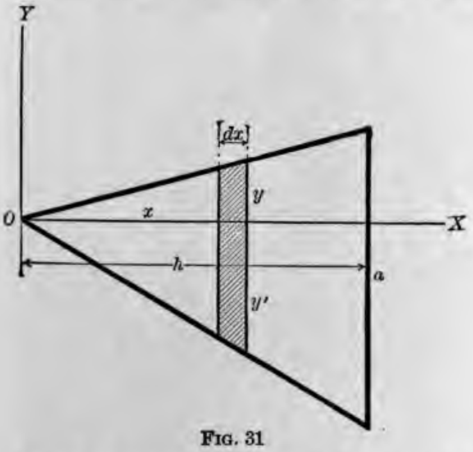


FIG. 31

Problem 21. Find the center of gravity of a triangle whose altitude is h and whose base is a. Take the origin at the vertex and draw the x-axis perpendicular to the base. (See Fig. 31.)

$$\bar{x} = \frac{\int x\,dF}{\int dF}.$$ Here $dF = (y + y')dx$, and from similar triangles

$$y + y' = \frac{a}{h}x, \text{ so that } dF = \frac{a}{h}x\,dx,$$

and
$$\bar{x} = \frac{\frac{a}{h}\int_0^h x^2dx}{\frac{a}{h}\int_0^h xdx} = \frac{\left.\frac{x^3}{3}\right]_0^h}{\left.\frac{x^2}{2}\right]_0^h} = \frac{2}{3}h.$$

The center of gravity is $\frac{2}{3}$ the distance from the vertex to the base, and since the *median* is a line of symmetry, it is a point on the median. It is, in fact, the point where the medians of the triangle intersect.

Problem 22. Find the center of gravity of a section of a flat ring, outside radius R_1 and inside radius R_2. (See Fig. 32.) Let the angle of the sector be 2θ. Take the origin at the center and let the x-axis bisect the angle 2θ.

$$\bar{x} = \frac{\int xdF}{\int dF} \text{ here, } dF = \rho d\rho d\alpha, \text{ and } x = \rho\cos\alpha,$$

so that
$$\bar{x} = \frac{\int\int \cos\alpha\, d\alpha \cdot \rho^2 d\rho}{\int\int \rho d\rho d\alpha}.$$

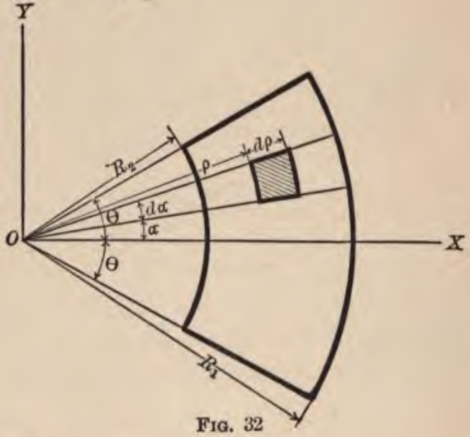


FIG. 32

Integrating the numerator first,

$$\int_{-\theta}^{+\theta}\cos\alpha d\alpha\int_{R_2}^{R_1}\rho^2 d\rho = \frac{R_1^3 - R_2^3}{3}\int_{-\theta}^{+\theta}\cos\alpha d\alpha = \frac{R_1^3 - R_2^3}{3}(2\sin\theta).$$

Integrating the denominator,

$$\int_{-\theta}^{+\theta} d\alpha \int_{R_2}^{R_1} \rho\, d\rho = \frac{R_1^2 - R_2^2}{2} \int_{-\theta}^{+\theta} d\alpha = (R_1^2 - R_2^2)\,\theta.$$

Therefore,
$$\bar{x} = \frac{2}{3} \frac{R_1^3 - R_2^3}{R_1^2 - R_2^2} \frac{\sin \theta}{\theta}.$$

If $R_2 = 0$, the sector becomes the sector of a circle, and $\bar{x}$ becomes

$$\bar{x} = \frac{2}{3} R_1 \frac{\sin \theta}{\theta}.$$

If the sector is a semicircle, that is, if $2\theta = \pi$, then, since $\theta = \frac{\pi}{2}$,

$$\bar{x} = \frac{2}{3} R_1 \left[\frac{1}{\frac{\pi}{2}} \right] = \frac{4 R_1}{3 \pi}.$$

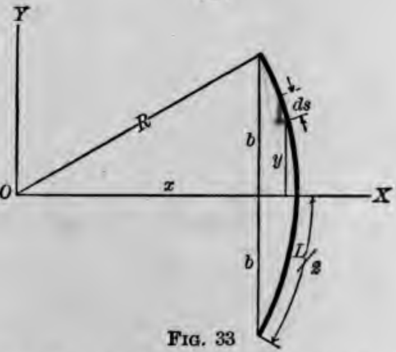


FIG. 33

Problem 23. Find the center of gravity of a portion of circular wire (Fig. 33) of length L and whose chord $= 2b$. Take the center of the circular arc as origin and let the x-axis bisect L. Then

$$\bar{x} = \frac{\int x\, ds}{\int ds}; \text{ but } R : x = ds : dy,$$

$$ds = \frac{R}{x} dy,$$

$$\bar{x} = \frac{R \int_{-b}^{+b} dy}{L} = \frac{2Rb}{L} = \frac{\textbf{radius} \times \textbf{chord}}{\textbf{arc}}.$$

For a semicircular wire $\bar{x} = \frac{\text{Diameter}}{\pi}$.

Problem 24. Find the center of gravity of a paraboloid of revolution. The equation of the generating curve being $y^2 = 2px$, and the greatest value of x, is a.

$$\bar{x} = \frac{2}{3}a.$$

Note. Use the same method as that used for the right circular cone.

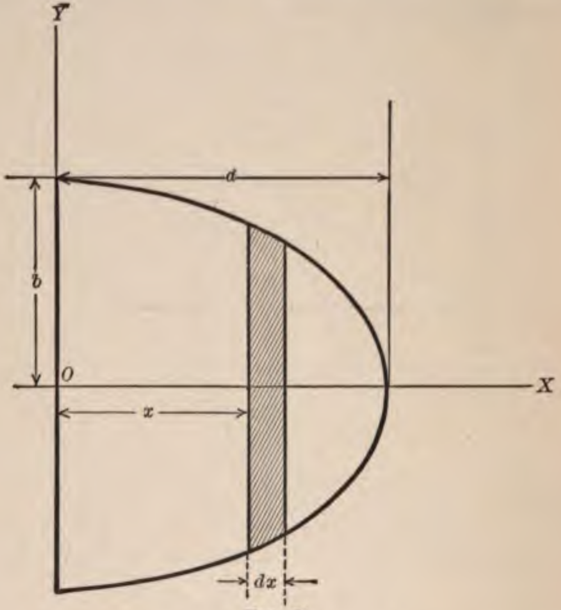


Fig. 34

Problem 25. Find the center of gravity of a semi-ellipse (Fig. 34) whose equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \qquad \bar{x} = \frac{\int x dF}{\int dF}$$

where $$dF = 2y dx = 2\frac{b}{a}\sqrt{a^2 - x^2}\, dx,$$

therefore
$$\bar{x} = \frac{2\frac{b}{a}\int_0^a x\sqrt{a^2-x^2}\,dx}{2\frac{b}{a}\int_0^a \sqrt{a^2-x^2}\,dx} = \frac{-\frac{1}{3}(a^2-x^2)^{\frac{3}{2}}\Big]_0^a}{\frac{1}{2}\Big[x\sqrt{a^2-x^2}+a^2\sin^{-1}\frac{x}{a}\Big]_0^a}$$

$$= \frac{\frac{a^3}{3}}{\frac{a^2}{2}\cdot\frac{\pi}{2}} = \mathbf{\frac{4a}{3\pi}}.$$

Problem 26. Find the center of gravity of a hemisphere, the radius of the sphere being r. Let the equation of the generating circle of the surface be $x^2+y^2=r^2$. Then

$$\bar{x} = \frac{\int x\,dP}{\int dP}, \text{ where } dP = \gamma\pi y^2 dx = \gamma\pi(r^2-x^2)dx,$$

$$\int_0^r \gamma\pi x(r^2-x^2)dx = \gamma\pi\Big[r^2\frac{x^2}{2}-\frac{x^4}{4}\Big]_0^r = \frac{\gamma\pi r^4}{4},$$

and
$$\int_0^r \gamma\pi(r^2-x^2)dx = \gamma\pi\Big[rx-\frac{x^3}{3}\Big]_0^r = \frac{2\gamma\pi r^3}{3}.$$

$$\bar{x} = \frac{\frac{\gamma\pi r^4}{4}}{\frac{2\gamma\pi r^3}{3}} = \mathbf{\frac{3}{8}r}.$$

Problem 27. Find the center of gravity of the area between the parabola, the y-axis, and the line AB in Problem 20.

Problem 28. A quadrant of a circle is taken from a square whose sides equal the radius of the circle. (See Fig. 35.) Find the center of gravity of the remaining area.

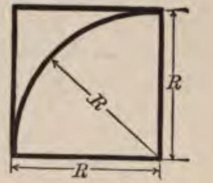


FIG. 35

Problem 29. Suppose that the corners A and C of the angle iron in Fig. 22 are cut to the arc of a circle of $\frac{5}{16}$ in. radius and the angle at B is filled to the arc of a circle of $\frac{3}{8}$ in. radius; what would be the change in $\bar{x}$ and $\bar{y}$?

Problem 30. Show that the center of gravity of the segment of a circle (Fig. 36), included between the arc $2s$ and the chord $2d$, is given by $\bar{x} = \frac{8d^3}{12F}$, where F is the area of the segment.

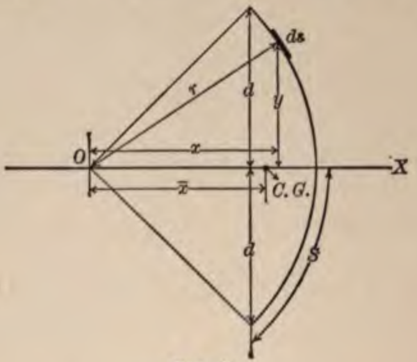


FIG. 36

25. Center of Gravity of Counterbalance of Locomotive Drive Wheel.

— In Fig. 37 the drive wheel is indicated by the circle and the counterbalance by the portion inclosed by the heavy lines, the point O is the center of the wheel, and α is the angle subtended by the counterbalance. The point O' is the center of the circle forming the inner boundary of the counterbalance, and β is the angle subtended by the counterbalance at this point. Let F_1 represent the area of the segment of radius r and F_2 the area

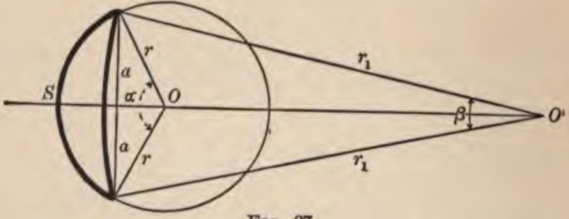


FIG. 37

of the segment of radius r_1. Also let x_1 represent the distance of the center of gravity of F_1 from O, and x_2' the distance of the center of gravity of F_2 from O'. Then, from Problem 30,

$$x_1 = \frac{8\,a^3}{12 F_1} \text{ and } x_2' = \frac{8\,a^3}{12\,F_2}.$$

But x_2, the distance of the center of gravity of F_2 from O,

$$= x_2' - OO' = \frac{8\,a^3}{12\,F_2} - \left(r_1 \cos\frac{\beta}{2} - r\cos\frac{\alpha}{2}\right).$$

So that x, the distance of the center of gravity of the counterbalance from O, equals $\frac{F_1 x_1 - F_2 x_2}{F_1 - F_2}$. It is seen that F_1 equals area of sector minus area of triangle equals $\frac{\alpha r^2}{2} - ar\cos\frac{\alpha}{2}$, and similarly,

$$F_2 = \frac{\beta r_1^{\,2}}{2} - ar_1\cos\frac{\beta}{2}.$$

Therefore,
$$\bar{x} = \frac{F_2\left(r_1\cos\frac{\beta}{2} - r\cos\frac{\alpha}{2}\right)}{F_1 - F_2}.$$

26. Simpson's Rule. — When the algebraic equation of a curve is known, it is expressed as $y = f(x)$, and the area between the curve and either axis is always determined by integration. In Fig. 38 the area $ABCD$ is expressed by the integral

$$\int_a^b y\,dx = \int_a^b f(x)\,dx,$$

when the curve represented by $y = f(x)$ is continuous between A and B.

In many engineering problems the curve is such that its equation is not known, so that approximate methods of obtaining the areas under the curve must be resorted to. One of these methods of approximation is known as *Simpson's Rule*. Suppose the curve in question is the curve AB (Fig. 39) and it is desired to find the area between the portion AB and the x-axis. Divide the length $b-a$ into an even number of equal parts n (here $n=10$). Consider the portion $CDEF$ and imagine it magnified as shown in Fig. 40. Pass a parabolic arc through the points C, D, G; then the area $CDEF$ is approximated by the area of the parabolic segment $CGDI$ plus the area of the trapezoid $CDEF$, therefore area $CGDEF$ $= \frac{1}{2}(y_2 + y_4)EF + \frac{2}{3}(y_3 - \frac{1}{2}[y_2 + y_4])EF$, since the area of

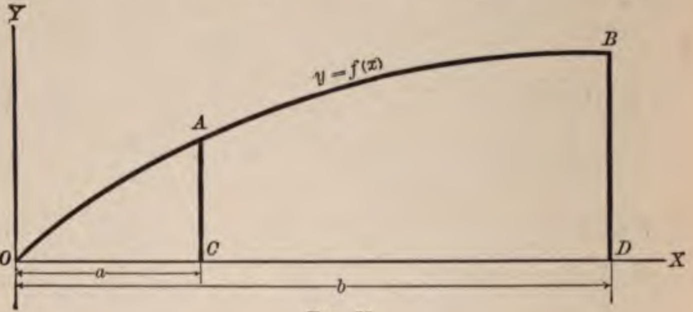


FIG. 38

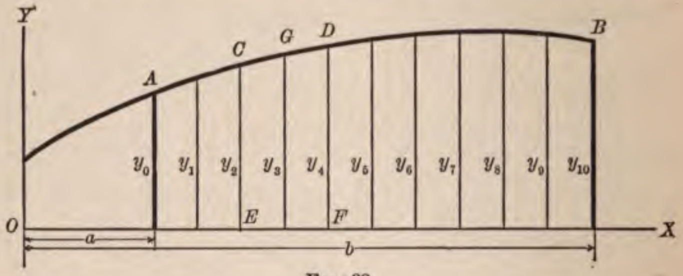


FIG. 39

the parabolic segment is $\frac{2}{3}$ the area of the circumscribing parallelogram. Since $EH = \dfrac{b-a}{n} = \Delta x$, this area may be written

$$\tfrac{1}{3}\Delta x(y_2 + 4y_3 + y_4).$$

In a similar way the next two strips to the right will have an area, $\dfrac{\Delta x}{3}(y_4 + 4y_5 + y_6)$, and the next two strips, an area, $\dfrac{\Delta x}{3}(y_6 + 4y_7 + y_8)$, and so on. Adding all these so as to get the total area under the portion of the curve AB, we get

$$\text{total area} = \frac{b-a}{3 \cdot 10}[y_0 + 4(y_1 + y_3 + y_5 + y_7 + y_9) + 2(y_2 + y_4 + y_6 + y_8) + y_{10}],$$

or in general for n divisions,

$$\text{total area} = \frac{b-a}{3n}[y_0 + 4(y_1 + y_3 + y_5 + \cdots y_{n-1}) + 2(y_2 + y_4 + y_6 + \cdots y_{n-2}) + y_n],$$

and this is Simpson's formula for determining approximately the area under a curve. It is easy to see that the smaller Δx, the less the approximation will be.

27. Application of Simpson's Rule. — Simpson's Rule may be made use of in determining approximately not only areas, but volumes and moments. On account of its use in adding moments Simpson's formula may be employed in finding the center of gravity of areas or volumes bounded by lines or surfaces whose equations are not known. Suppose, for example, it is desired to

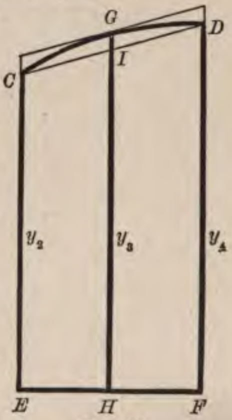


FIG. 40

know the volume and position of the center of gravity of a coal bunker of a ship as shown in Fig. 41. The bunker is 80 ft. long and the areas A_0, A_1, A_2, A_3, A_4, are as follows:

$$A_0 = 400 \text{ sq. ft.},$$
$$A_1 = 700 \text{ sq. ft.},$$
$$A_2 = 650 \text{ sq. ft.},$$
$$A_3 = 600 \text{ sq. ft.},$$
$$A_4 = 400 \text{ sq. ft.}$$

The distance between the successive areas is 20 ft. Applying Simpson's formula for volume,

$$\text{volume} = \frac{80}{(3)(4)}[A_0 + 4(A_1 + A_3) + 2\,A_2 + A_4].$$

Summing the values A_0x_0, A_1x_1, A_2x_2, etc., we obtain

$$\Sigma vx = \frac{80}{(3)(4)}[A_0x_0 + 4(A_1x_1 + A_3x_3) + 2\,A_2x_2 + A_4x_4],$$

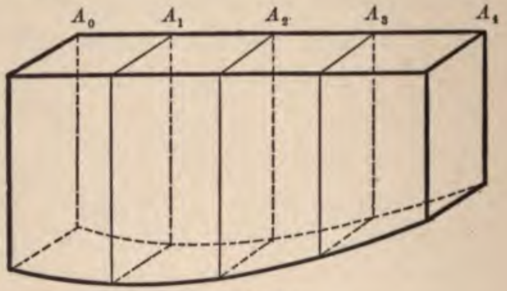


FIG. 41

where $x_0 = 0$, $x_1 = 20$, $x_2 = 40$, $x_3 = 60$, $x_4 = 80$. The position of the center of gravity from the fore end can now be obtained from the relation

$$\bar{x} = \frac{\Sigma vx}{v}.$$

A value of $\bar{x}$ might also have been obtained from the formula

$$\bar{x} = \frac{v_0x_0 + v_1x_1 + v_2x_2 + v_3x_3 + v_4x_4}{v_0 + v_1 + v_2 + x_3 + v_4}$$

by simply adding the terms in the numerator and denominator. Compare the value obtained by using this formula with that obtained by using Simpson's formula.

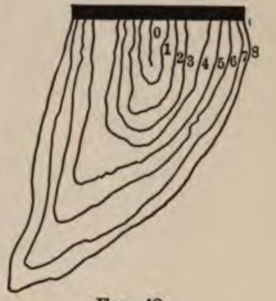


FIG. 42

Problem 31. A reservoir with five-foot contour lines is shown in Fig. 42. Find the volume of water and the distance of the center of gravity from the surface of the water, if the areas of the contour lines are as follows: $A_0 = 0$, $A_1 = 100$ sq. ft., $A_2 = 200$ sq. ft., $A_3 = 500$ sq. ft., $A_4 = 600$ sq. ft., $A_5 = 1000$ sq. ft., $A_6 = 1500$ sq. ft., $A_7 = 2000$ sq. ft., $A_8 = 2500$ sq. ft. Making substitutions in the Simpson's formula, it becomes, for the volume,

$$\text{volume} = \frac{35}{(3)(8)}\left[A_0 + 4(A_1 + A_3 + A_5 + A_7) + 2(A_2 + A_4 + A_6) + A_8\right].$$

Summing the values A_0x_0, A_1x_1, A_2x_2, etc., by Simpson's formula, we have

$$\Sigma vx = \frac{35}{(3)(8)}\left[A_0x_0 + 4(A_1x_1 + A_3x_3 + A_5x_5 + A_7x_7) + 2(A_2x_2 + A_4x_4 + A_6x_6) + A_8x_8\right],$$

where $x_0 = 0$ ft., $x_1 = 5$ ft., $x_2 = 10$ ft., etc., so that

$$\bar{x} = \frac{\Sigma vx}{v}.$$

Both numerator and denominator are computed by Simpson's formula.

Compute $\bar{x}$ by means of the formula,

$$\bar{x} = \frac{v_0x_0 + v_1x_1 + v_2x_2 + v_3x_3 + \text{etc.}}{v_0 + v_1 + v_2 + v_3 + \text{etc.}},$$

and compare with previous result.

Problem 32. Compute $\bar{x}$ for the parabolic area of Fig. 30, by using Simpson's Rule, and compare the result with that obtained by integration.

Problem 33. By Simpson's Rule find the area and center of gravity of the rail section shown in Fig. 43.

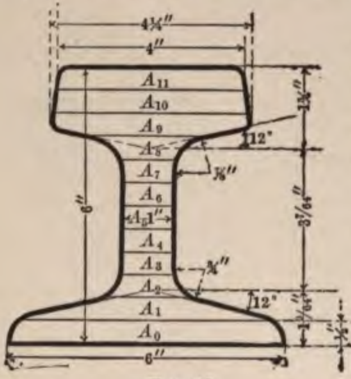


FIG. 43

$A_{11} = 2.05$ $A_7 = .49$ $A_3 = .55$
$A_{10} = 2.07$ $A_6 = .51$ $A_2 = .61$
$A_9 = 1.89$ $A_5 = .51$ $A_1 = 1.95$
$A_8 = .82$ $A_4 = .51$ $A_0 = 2.95$

and the horizontal distances are as follows:

$u_{12} = 4''$ $u_8 = 1.24''$ $u_4 = 1.0''$
$u_{11} = 4.08''$ $u_7 = 1.18''$ $u_3 = 1.24''$
$u_{10} = 4.24''$ $u_6 = 1.0''$ $u_2 = 2.23''$
$u_9 = 2.5''$ $u_5 = 1.0''$ $u_1 = 5.5''$
$u_0 = 6''$

Problem 34. Find the center of gravity of the deck beam section shown in Fig. 44. Use the formula

$$\bar{x} = \frac{F_1x_1 + F_2x_2 + F_3x_3 + \text{etc.}}{F_1 + F_2 + F_3 + \text{etc.}},$$

and divide the bulb area into convenient areas, say F_5, F_6, F_7, etc. Check the result thus obtained with that obtained by balancing a stiff paper model over a knife edge.

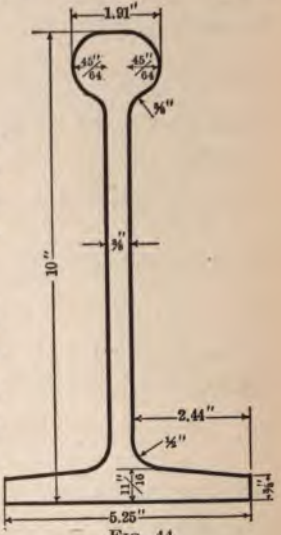


FIG. 44

28. Durand's Rule.— A method of finding the area of irregular areas was published by Professor Durand in the *Engineering News*, Jan. 18, 1894. The rule states that the total area of an irregular curve equals

$$\left(\frac{b-a}{n}\right)[(1-0.6)u_0+(1+0.1)u_1+u_2+u_3+u_4+u_5 \\ +\cdots(1+0.1)u_{n-1}+(1-0.6)u_n],$$

where the u's have the same meaning as before. The divisions may be even or odd. The student is advised to make use of this rule as well as Simpson's Rule and compare the results.

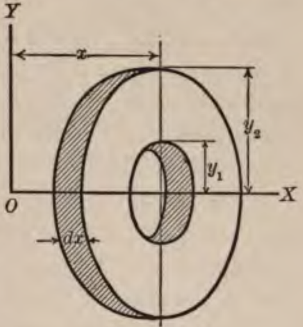


FIG. 45

29. Theorems of Pappus and Guldinus. — Let the disk in Fig. 45 be any slice cut from a solid of revolution by two parallel planes perpendicular to the axis of revolution and at a distance dx apart. The volume of this slice is $dv=\pi(y_2^2-y_1^2)dx$, so that the volume of the whole solid is $v=\pi\int(y_2^2-y_1^2)dx$. The generating figure of this slice, dF, equals $(y_2-y_1)dx$, and the distance of its center of gravity from the x-axis is $\frac{y_2+y_1}{2}$. We have seen that

$$\bar{y}=\frac{\int y\,dF}{\int dF};$$

then

$$\bar{y}=\frac{1}{F}\int y\,dF=\frac{1}{F}\int\left(\frac{y_2+y_1}{2}\right)(y_2-y_1)dx=\frac{1}{2F}\int(y_2^2-y_1^2)dx,$$

and this, considering the expression for volume, becomes

$$\bar{y}=\frac{1}{2\pi F}v;$$

therefore,

$$v=2\pi\bar{y}F.$$

This may be stated as a general principle as follows:

The volume of any solid of revolution is equal to the area of the generating figure times the distance its center of gravity moves.

Problem 35. Find the volume of a sphere, radius r, by the above method, assuming it to be generated by a semicircular area revolving about a diameter.

Problem 36. Assuming the volume of the sphere known, find the center of gravity of the generating semicircular area.

Problem 37. Find the volume of a right circular cone, assuming that the generating triangle has a base r and altitude h.

Problem 38. Assuming the volume of the cone known, find the center of gravity of the generating triangle.

Problem 39. The parabolic area of Problem 20 revolves about the x-axis; find the volume of the resulting solid.

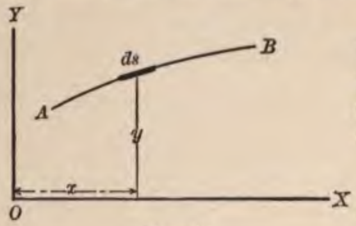


FIG. 46

Problem 40. Find the volume of an anchor ring, if the radius of the generating figure is a and the distance of its center from the axis of revolution is r.

Let the curve AB (Fig. 46), of length l, be the generating curve of a surface of revolution. The area of the surface generated by ds will be $dF = 2\pi y ds$, and the area of the whole surface will be $F = 2\pi \int y ds$. The center of gravity of this curve AB is given by the expression

$$\bar{y} = \frac{\int y ds}{\int ds} = \frac{1}{l}\int y ds;$$

$$\bar{y} = \frac{F}{2\pi l}, \text{ or } F = 2\pi \bar{y} l.$$

This may be stated as follows: *The area of any surface of revolution is equal to the length of the generating curve times the distance its center of gravity moves.*

Problem 41. Find the surface of a sphere, radius r, assuming the generating line to be a semicircular arc.

Problem 42. Find the center of gravity of a quadrant of a circular wire, radius of the circle r; use results obtained above.

Problem 43. Find the surface of the paraboloid in Problem 39.

CHAPTER V

COUPLES

30. Couples Defined. — In Art. 20, Case (*c*), it was shown that the resultant of two parallel forces in a plane was equal to the algebraic sum of the two forces. The consideration of the case when the forces are equal and opposite in direction, that is, where the resultant is zero, will now be considered. It is easy to see that since the resultant is zero, the two forces tend to produce only a rotation of the rigid body about a gravity axis perpendicular to the plane of the forces. Such a pair of equal and opposite parallel forces is called a *couple*. Let it be represented as in Fig. 47 (a), the two forces being P, and d the distance between the lines of action of the forces. This distance d is called the *arm* of the couple; one of the forces times

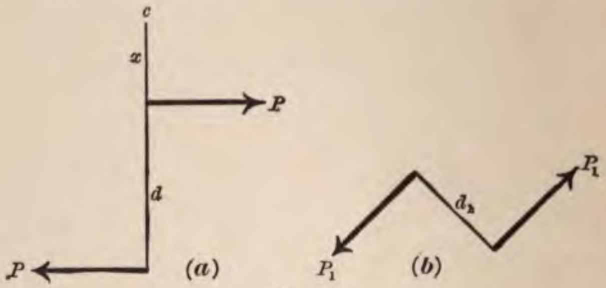


FIG. 47

the arm is called the *moment* of the couple. It was found in Art. 20 that the algebraic sum of the moments of the resultant and the system of parallel forces with respect to any point in their plane, is zero. In this case, since the resultant is zero, the moment of the forces of the couple with respect to any point in the plane is equal to the sum of the moments of the two forces with respect to that point. Let the point be C, Fig. 47 (a), distant x from the force P; then $-Px-P(-d-x)$ represents the sum of the moments of the two forces with respect to the point C (calling distance below C negative). This sum is equal to Pd, the moment of the couple. The student should take C in different positions and show that the moment of the two forces with respect to any point in the plane is always Pd. Since the moment consists of force times distance, it is measured in terms of the units of force and distance; that is, foot-pounds or inch-pounds, usually. If the couple tends to produce rotation in the clockwise direction, the moment is said to be *negative;* and if counter-clockwise, *positive.*

31. Representation of Couples. — The couple involves magnitude (moment) and direction (rotation), and may, therefore, be represented by an arrow, the length of the line being proportional to the moment of the couple, and the arrow indicating the direction of rotation. In order to make the matter of direction of rotation clear, the agreement is made that the arrow be drawn perpendicular to the plane of the couple on that side from which the rotation appears counter-clockwise. This means that if we look along the arrow pointing toward us, the rotation appears counter-clockwise. Thus, the couple of Fig. 47 (*a*), whose moment is Pd, may be represented by the arrow in Fig. 48 (*a*), where the length of line AB is proportional to Pd and the couple is in a plane through B and perpendicular to AB. The line AB is sometimes called the axis of the couple; it may be drawn perpendicular to the plane of the couple at any point in that plane, since the moment is constant for any point in the plane. In a similar way, the couple (*b*) in Fig. 47 whose moment is P_1d_1 is represented completely by the arrow (*b*), Fig. 48, the length CD being proportional to P_1d_1.

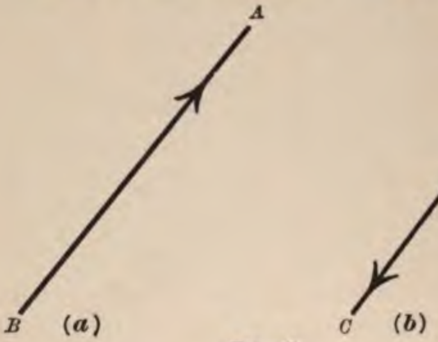


FIG. 48

NOTE. The arrows are placed slightly away from the ends, so that the moment arrows may not be confused with force arrows. These arrows, like force arrows, may be added algebraically when parallel, resolved into components and compounded into resultants; the principle of transmissibility holds and also the triangle and polygon laws as seen for force arrows. Several important conclusions follow easily as a result of this arrow representation.

Since a moment arrow represents both force and distance and direction of rotation, it is evident that it cannot be balanced by a single force arrow even though they have the same line of action and are opposite in direction. Hence, we conclude that *a single force cannot balance a couple.*

32. Couples in One Plane. — If the couples are all in the same plane, their moment arrows are all parallel, and may be added algebraically, so that the *resultant couple lies in the same plane and its moment is the algebraic sum of the moments of the individual couples.*

For example, in Fig. 49, the couples P_1d_1, P_2d_2, P_3d_3, P_4d_4, P_5d_5, P_6d_6, are all in the plane (ab); their resultant couple must also be in this plane, and its moment must be equal to the algebraic sum of the moments of these couples.

It is evident since the above is true that a couple may be transferred to any part of its plane without changing its effect upon the rigid body upon which it acts. This means, when applied to some particular rigid body, as a closed book, that the effect of a couple acting in the plane of one of the covers of the book (book remains closed) tends to produce rotation about an axis, perpendicular to the cover through the center of gravity of the book; and that this rotation is the same no matter where the couple acts, provided it remains always in the same plane. The moment arrow of the resultant couple will be perpendicular to the cover of the book and on the side from which the rotation appears counter-clockwise. The student should endeavor to see the application of the above theorem and to see that it agrees with his observations.

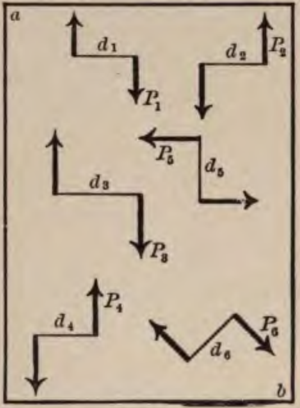


FIG. 49

33. Couples in Parallel Planes. — The moment arrow represents a couple in magnitude and direction of rotation and shows that the plane of the couple is perpendicular to its line. This moment arrow represents any couple of *given moment* and *direction* in any plane perpendicular to its line. It is evident, then, *that a couple may be transferred to any parallel plane without changing its effect upon the rigid body upon which it acts.* Applied to the case of the book in the preceding article, it may be said that the effect of the couple would be unchanged if it acted in the plane of the other cover or in any of the leaves.

34. Couples in Intersecting Planes. — Suppose all the couples in a plane (1, 2) be added and let AB (*a*) (Fig. 48) represent the moment arrow of the resultant couple, and let the sum of all the couples in the plane (2, 3) intersecting (1, 2) be represented by CD (*b*) (Fig. 50). These moment arrows are perpendicular to their respective planes and may be moved about without changing the effect of the couple. Move A and C to some point on the line of intersection of the two planes. The resultant moment arrow is now found by the parallelogram law. The resultant couple has a moment represented by AE and acts in a plane perpendicular to AE and making an angle α with the plane (2, 3).

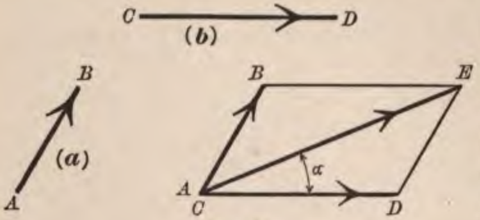


FIG. 50

Problem 44. Two forces, each equal 10 lb., act in a vertical plane so as to form a positive couple. The distance between the forces is 2 ft.; another couple whose moment is equal to 20 in.-lb. acts in a horizontal plane and is negative. Required the resultant couple, its plane, and direction of rotation.

Problem 45. A couple whose moment is 10 ft.-lb. acts in the xy-plane; another couple whose moment is -30 in.-lb. acts in the xz-plane, and another couple whose moment is $-$ 25 ft. in-lb. acts in the yz-plane. Required the amount, direction, and location of the resultant couple that will hold these couples in equilibrium. (x, y, and z-axes are at right angles with each other in this case.)

CHAPTER VI

NON-CONCURRENT FORCES

35. Forces in a Plane. — The most general case of forces in a plane is that one in which the forces are non-concurrent and non-parallel. We shall now consider such a case. Let the forces be P_1, P_2, P_3, P_4, etc., as shown in Fig. 51, and let them have the directions shown. For the sake of analysis, introduce at the origin two equal and opposite forces P_1, parallel to P_1, two equal and oppo-

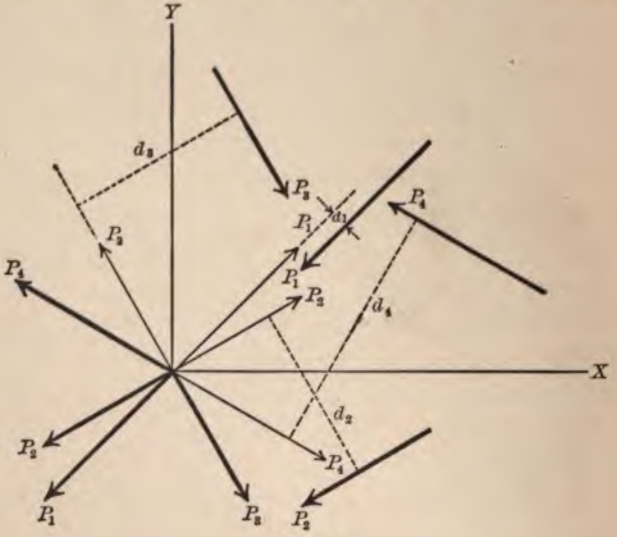


Fig. 51

site forces P_2, parallel to P_2, and so on for each force. The introduction of these equal and opposite forces at the origin cannot change the state of motion of the rigid body.

The original force P_1 taken with one of the forces P_1, introduced at the origin, forms a couple whose moment is P_1d_1. The same is also true of the forces P_2, P_3, P_4, etc., giving respectively moments P_2d_2, P_3d_3, P_4d_4, etc. In addition to these couples there is a system of concurring forces at the origin P_1, P_2, P_3, P_4, etc. The resultant of this system is, as has been shown (Art. 16),

$$R = \sqrt{(\Sigma x)^2 + (\Sigma y)^2}.$$

The moments P_1d_1, P_2d_2, P_3d_3, etc., being all in one plane, may be added algebraically (see Art. 32), giving the moment of the resultant couple as ΣPd.

The system of non-concurrent forces in a plane may be reduced, then, to a single force R at the origin (arbitrarily selected) and a single couple whose plane is the plane of the forces.

For equilibrium, $\boldsymbol{R = 0}$ and $\boldsymbol{\Sigma Pd = 0}$, or $\boldsymbol{\Sigma x = 0}$, $\boldsymbol{\Sigma y = 0}$, and $\boldsymbol{\Sigma Pd = 0}$; that is, for equilibrium, the sum of the components of the forces along each of the two axes is zero and the sum of the moments with respect to any point in the plane is zero.

Considering as a special case the case where the forces are concurring, it is seen that Pd is always zero (see Art. 16). The case of a system of parallel forces in a plane may also be considered as a special case of the above. (Art. 20 and Art. 31).

Problem 46. The following forces act upon a rigid body: a force of 100 lb. whose line of action makes an angle of 45° with the horizontal, and whose distance from an arbitrarily selected origin is 2 ft.; also a force of 50 lb. whose line of action makes an angle of 120° with the horizontal, and whose distance from the origin is 3 ft.; and a force of 500 lb. whose line of action makes an angle of 300° with the horizontal and whose distance from the origin is 6 ft. Find the resultant force and the resultant couple.

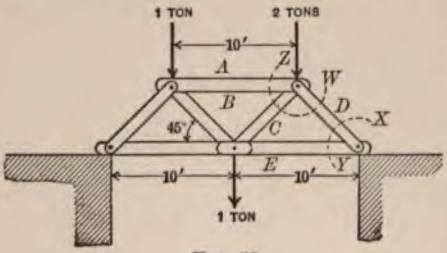


Fig. 52

Problem 47. It is required to find the stress in the members AB, BC, CD, and CE of the bridge truss shown in Fig. 52.

Note. The member AB is the member between A and B, the member CD is the member between C and D, etc. This is a type of Warren bridge truss. All pieces (members) are pin-connected so that only two forces act on each *member*. The members are, therefore, under simple tension or compression; that is, in each member the forces act along the piece. Usually, in such cases there are no loads on the upper pins.

Solution of Problem. The reactions of the supports are found by considering all the external forces acting on the truss. Taking moments about the left support, we get the reaction at the right support, equal 4500 lb. Summing the vertical forces or taking moments about the right-hand support, the reaction at the left-hand support is found to be 3500 lb.

Cutting the truss along xy and putting in the forces exerted by the left-hand portion, consider the right-hand portion (see Fig. 53). The forces C and T act along the pieces, forming a system of concurring forces. For equilibrium, then, $\Sigma x = 0$ and $\Sigma y = 0$, giving two equations, sufficient to determine the unknowns C and T. The forces in the members CD and CE may now be considered known.

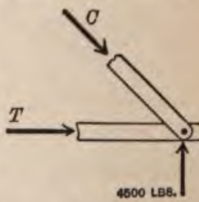


Fig. 53

Cutting the truss along the line ZW and putting in the forces exerted by the remaining portion of the truss, we have the portion represented in Fig. 54. This gives a system of concurring forces of which C and $2T$ are known, so that from the equations $\Sigma x = 0$ and $\Sigma y = 0$ the remaining forces d and e may be found.

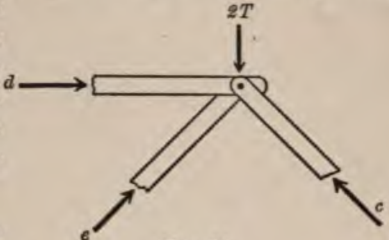


FIG. 54

Problem 48. Find the stress in each of the members AB and BC of the simple roof truss shown in Fig. 55. The truss is pin-connected, and all the members are under simple tension or compression except the horizontal piece, which is under flexure. The method of cutting the truss, employed in the preceding problem, cannot be employed to advantage here, since the stress in the horizontal piece is not along the piece. The simplest method of solution for such a case is to take the whole member in question and consider all of the forces acting.

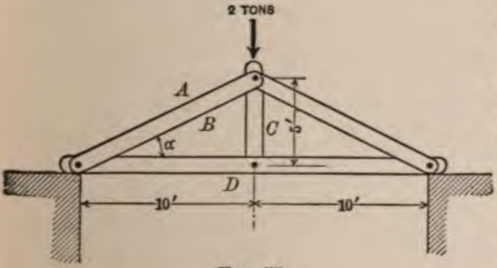


FIG. 55

In this case we have (see Fig. 56) a system of non-concurring forces in a plane. For equilibrium $\Sigma x = 0$ and $\Sigma y = 0$ and $\Sigma Pd = 0$, from which P, P, and P_1 may be determined.

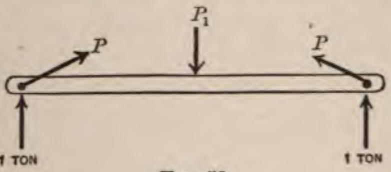


FIG. 56

Problem 49. In the crane shown in Fig. 57 (a) find the forces acting on the pins and the tension in the tie AC. The method of cutting cannot be used in this case since the vertical and horizontal members are in flexure. Taking the horizontal member and considering all of the forces acting upon it, we have the system of non-concurring forces shown in Fig. 57 (*b*). Three unknowns are involved, P_3, P_1, and P_2, and these may be determined by three equations $\Sigma x = 0$, $\Sigma y = 0$, and $\Sigma Pd = 0$. It is to be remembered that the pin pressure at *E* is unknown in magnitude and direction. In all such cases it is usually more convenient to resolve this unknown pressure into its vertical and horizontal components, giving two unknown forces in known directions instead of one unknown force in an unknown direction. This will be done in all problems given here. In the present case the two forces P_1 and P_2 are the components of the unknown pin pressure.

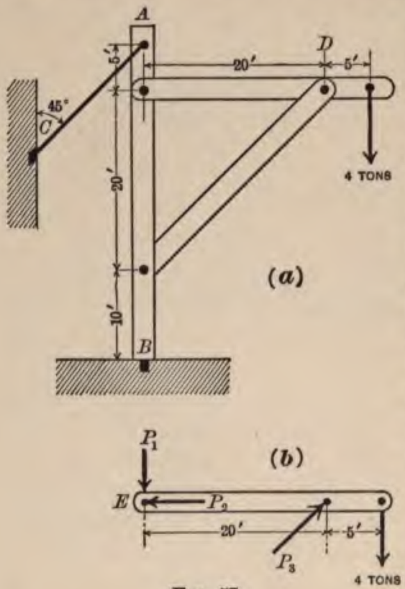


FIG. 57

The tension in the tie *AC* may be found by considering the forces acting on the whole crane and taking moments about *B*. Thus $\Sigma Pd = 0$ gives, calling the tension in the tie *T*,

$$T 35 \sin 45^\circ = 8000\ (25),$$

or

$$T = \frac{8000\ (25)}{35 \sin 45^\circ}.$$

Problem 50. In the crane shown in Fig. 58 (*a*) find the tension in the ties *T* and *T*′ and the compression in the boom. The method of cutting may be used here to determine the tension *T* and the compression in the boom, since *AB* is not in flexure, if we neglect its own weight. Cutting the structure about the point *A* and drawing the forces acting on the body, we have the system shown in Fig. 58 *b*.

The forces W may be considered as acting at the center of the pulley. The system of forces is concurring, so that $\Sigma x = 0$ and $\Sigma y = 0$ are sufficient to determine T and C. T' may be found by considering the forces acting on the whole crane and taking moments about the lowest point B.

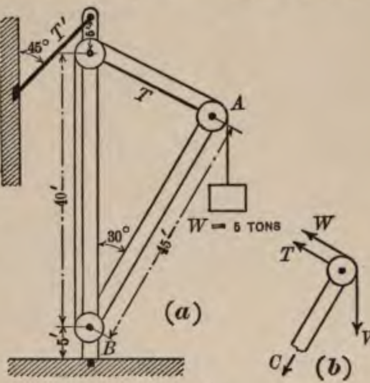


FIG. 58

NOTE. Neglecting friction, the tension W in the cord supporting the weight is transmitted undiminished throughout its length.

Problem 51. Find the horizontal and vertical components of the forces acting on the pins of the structure shown in Fig. 59.

SUGGESTION. First take the vertical strip and consider all the forces acting on it.

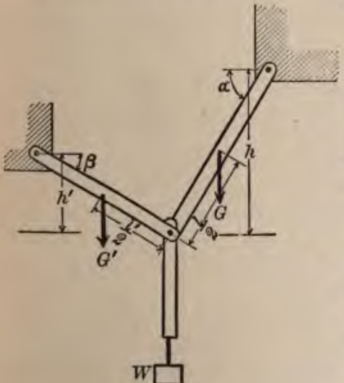


FIG. 59

Problem 52. Find the forces acting on the pins of the structure shown in Fig. 60, the weight of the members AD, BF, and CE being 600 lb., 400 lb., and 100 lb., respectively.

Problem 53. A traction engine is passing over a bridge, and when it is in the position shown in Fig. 61 one half of the load is carried by each truss. The weight of the engine is transmitted by the floor beams to the cross beams, and these are carried at the pin connections of the truss. Find the stress in the members AB, BC, CE, CD, and DF, for the position of the engine shown.

Note. The floor beams are supposed to extend only from one cross beam to another.

Problem 54. In Problem 50, suppose the weight of the boom to be one ton; find the tensions T and T' and the pin pressures.

Note. The boom is now under flexure, so that the method of cutting cannot be made use of.

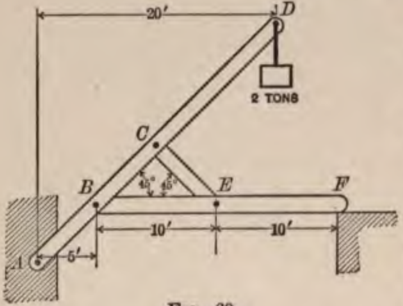


Fig. 60

Problem 55. A dredge or steam shovel, shown in outline in Fig. 62, has a dipper with capacity of 10 tons. When the boom and dipper are in the position shown, find the forces acting on AB, CD, and EF.

Suggestion. Consider first all the forces acting on CD, then all the forces acting on AB.

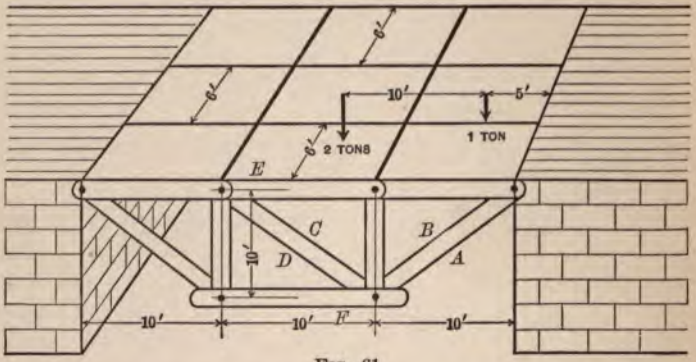


Fig. 61

Note. The member EF has been introduced as such for the sake of analysis; it replaces two legs, forming an A frame. The projection of the point F is 6 ft. from the point E.

Problem 56. Suppose the members of the structure in Problem

55 to have weights as follows: *AB*, 15 tons, and *CD*, 3 tons, not including the 10 tons of dipper and load. Find the stresses as required in preceding problem.

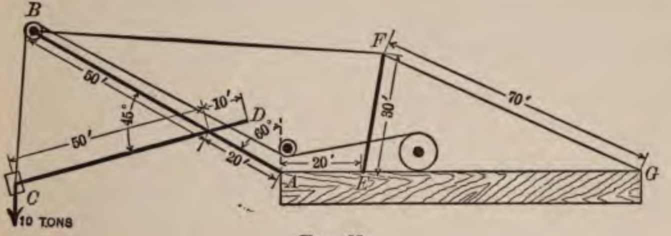


FIG. 62

Problem 57. Suppose the beam in Problem 5 to be 20 ft. long and to have a weight of 2000 lb.; find the pin reaction and the tension in the tie.

Problem 58. Assume that the compression members of the Warren bridge truss of Problem 47 have each a weight of 500 lb.; find the stress in the members *BC* and *CE*.

36. Forces in Space, Non-intersecting and Non-parallel. — If a rigid body is acted upon by any system of forces in space P_1, P_2, P_3, P_4, P_5, P_6, etc., making angles with the arbitrarily chosen axes, x, y, and z, α_1, β_1, γ_1; α_2, β_2, γ_2; α_3, β_3, γ_3, etc. (see Fig. 63), *it can be shown that the system may be replaced by a single force and a single couple.* The single force acts at the origin, and its direction angles are α, β, and γ. The resultant couple acts in a plane whose direction angles are λ, μ, ν.

Introduce at the origin two equal and opposite forces P_1 parallel to the line of action of P_1. One of these forces P_1 together with the original P_1 form a couple whose moment is $P_1 d_1$, and this couple may be represented by a moment arrow at the origin, perpendicular to the plane of the couple (see Art. 31). We thus have replaced the force P_1 by an equal and parallel force at the origin and a couple represented at the origin by a moment arrow P_1d_1. Proceeding in the same way with P_2, P_3, P_4, etc., we finally have instead of the original system

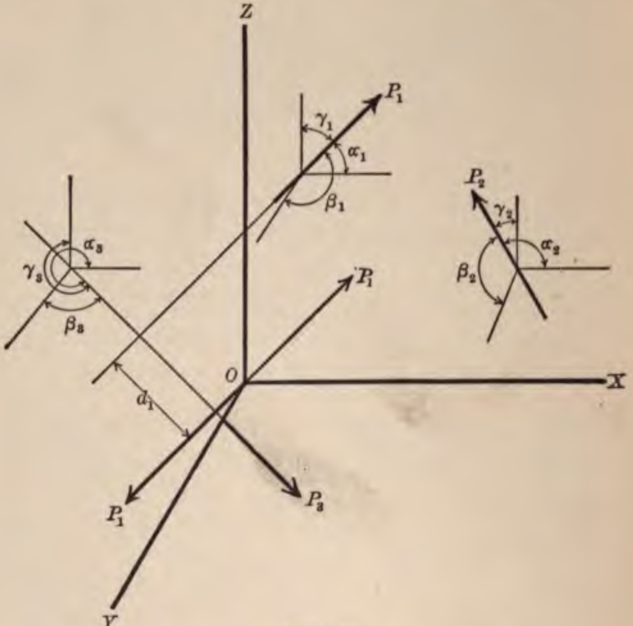


FIG. 63

of non-concurring, non-parallel forces in space, a system of concurring forces P_1, P_2, P_3, P_4, etc., at the origin and a system of moment arrows P_1d_1, P_2d_2, P_3d_3, etc., represented at the origin. The forces may be combined into a single resultant as in Art. 17, and we then have

$$R = \sqrt{\Sigma x^2 + \Sigma y^2 + \Sigma z^2},$$

whose direction cosines are

$$\cos \alpha = \frac{\Sigma x}{R}, \cos \beta = \frac{\Sigma y}{R}, \cos \gamma = \frac{\Sigma z}{R}.$$

Since the moment arrows also follow the same laws as the force arrows, we may also write the moment of the resultant couple,

$$M = \sqrt{M_x^2 + M_y^2 + M_z^2},$$

where

$$M_x = P_1 d_1 \cos \lambda_1 + P_2 d_2 \cos \lambda_2 + \text{etc.},$$
$$M_y = P_1 d_1 \cos \mu_1 + P_2 d_2 \cos \mu_2 + \text{etc.},$$
$$M_z = P_1 d_1 \cos \nu_1 + P_2 d_2 \cos \nu_2 + \text{etc.}$$

The direction angles of M are λ, μ, ν, and these are defined as follows (see Fig. 64):

$$\cos \lambda = \frac{M_x}{M}, \quad \cos \mu = \frac{M_y}{M}, \quad \cos \nu = \frac{M_z}{M}.$$

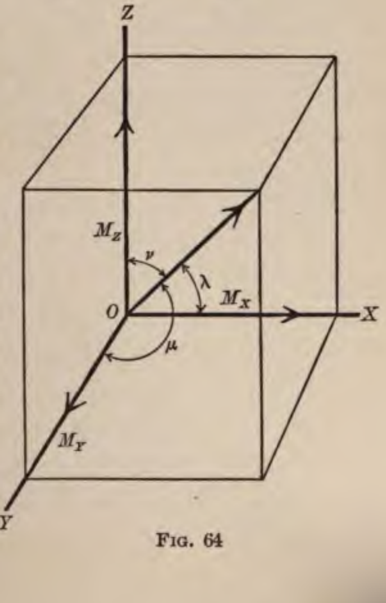


FIG. 64

This system of forces produces a translation of the body in the direction of R and a rotation about a gravity axis parallel to M. If $R = 0$ and $M \neq 0$, the body only rotates or is translated with uniform motion, and if $M = 0$ and $R \neq 0$, the body has only translation with possibly uniform rotation. For equilibrium both $R = 0$ and $M = 0$; that is, $\Sigma x = 0$, $\Sigma y = 0$, $\Sigma z = 0$, $M_x = 0$,

$M_y = 0$, and $M_z = 0$, or expressed in words: *the sum of the components of the forces along each of the three arbitrarily chosen axes is zero, and the sum of the moments with respect to each of these axes is zero.*

It may be further shown that the single force and resultant of the preceding article may be replaced by a single force and a couple whose plane of rotation is perpendicular to the line of action of the force.

Suppose M and R both drawn at the origin and let α be the angle between them. M may be resolved into components along and perpendicular to R, $G \cos \alpha$ along R, and $G \sin \alpha$ perpendicular to R. $G \sin \alpha$ may be replaced by another couple having the same moment. Let the forces be $-R$ and $+R$ and allow the $-R$ to act along the line of action of the resultant force. The other force of the couple acts along a line parallel to the direction of the resultant force. The forces $+R$ and $-R$ along the line of action of the resultant force neutralize each other, and we have left (*a*) a force, R, parallel to the original resultant, and (*b*) a couple, $G \cos \alpha$, acting in a plane perpendicular to R.

That is, the system reduces to a single force and a single couple whose plane is perpendicular to the line of action of the force, or we may say, *the effect of any system of forces acting on a rigid body, at any instant, is to cause an angular acceleration about the instantaneous axis of rotation and an acceleration of translation along that axis.* Such a system of forces is called a *screw wrench.* The instantaneous axis is called the central axis. This axis passes through the center of gravity of the body if the body is free to rotate.

Problem 59. A vertical shaft is acted upon by the belt pressures T_1 and T_2, the crank pin pressures P, and the reactions of the supports. See Fig. 65. Write down the six equations for equilibrium.

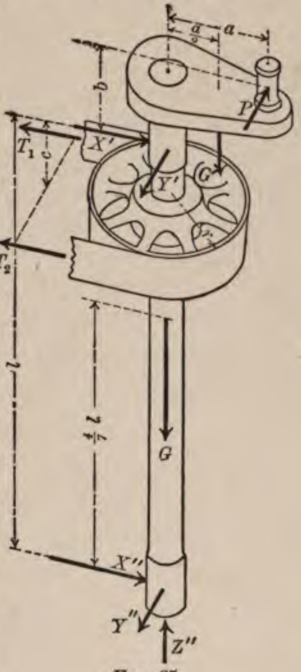


FIG. 65

NOTE. The y-axis has been chosen parallel to the force P, and T_1 and T_2 are parallel to the x-axis.

$$\Sigma x = x' + x'' - T_1 - T_2 = 0,$$
$$\Sigma y = y' + y'' - P = 0,$$
$$\Sigma z = z'' - G - G' = 0;$$
$$M_x = - Pb - y''l = 0,$$
$$M_y = x''l - T_1c - T_2c - G_1\frac{a}{2} = 0,$$
$$M_z = Pa + T_1r - T_2r = 0.$$

From these six equations six unknown quantities can be found. If G_1, G_2, T_1, and T_2 are known, the reaction of the supports and P may be found.

Problem 60. A crane shown in Fig. 66 has a boom 45 ft. long and a mast 30 ft. high. It is loaded with 20 tons, and the angle between the boom and mast is 45°. The two stiff legs each make angles of 30° with the mast and an angle of 90° with each other. Find the pin pressures in boom and mast, also the stress in

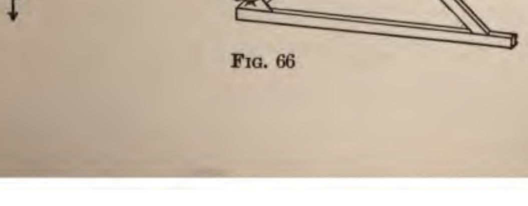


FIG. 66

the legs when (*a*) the plane of the crane bisects the angle between the legs, and (*b*) the plane of the crane makes an angle of 30° with one of them. If the boom weighs 4000 lb., find the stress in the legs when the plane of the crane bisects the angle between them. Assume that the pulleys A and B are at the ends of the boom and mast respectively.

Problem 61. Suppose the shaft of Problem 59 to be horizontal, find P and the reactions of the supports. Assume y horizontal and perpendicular to the shaft, and x vertical.

CHAPTER VII

MOMENT OF INERTIA

37. Definition of Moment of Inertia.—The study of many problems considered in mechanics brings to our attention the value of the integral of the form $\int y^2 dF$, where dF represents an area and y the distance of the center of gravity of that area from an axis of reference. A more general definition of *moment of inertia* would be the *product of an elementary area, mass, or volume by the square of its distance from a designated point, line, or plane.* The integral given above simply adds these products to give the moment of inertia of an entire area. In the case of a mass, the integral becomes $\int y^2 dM$, and of volume $\int y^2 dV$. If the area, mass, or volume is not continuous throughout, the limits of integration must be properly taken to account for the discontinuity. We shall designate moment of inertia by the letter I. Thus we write:

$$I = \int y^2 dF,$$

$$I = \int y^2 dM,$$

$$I = \int y^2 dV,$$

for area, mass, and volume, respectively. In finding the moment of inertia of several disconnected parts, it is often necessary to use the summation sign instead of the integral; we may then write:

$$I = \Sigma y^2 dF,$$

$$I = \Sigma y^2 dM,$$

$$I = \Sigma y^2 dV,$$

for area, mass, and volume, respectively.

Many problems that confront the engineer involve in their solution the consideration of the moment of inertia. This is the case when the energy of a rotating fly wheel, for example, is being determined. The energy of a rotating body (kinetic energy, Art. 133) is expressed as follows:

$$\text{Kinetic energy} = \frac{I\omega^2}{2},$$

where I is the moment of inertia with respect to the axis of rotation and ω is the angular velocity (see Art. 95). It is seen that the energy of rotating bodies, having the same angular velocity, or the same speed, is directly proportional to their moments of inertia. The quantity, therefore, plays a very important part in the consideration of rotating bodies.

In computing the strength of a beam or column it is necessary to consider the product of an area and the square of its distance from some line. This is called the moment of inertia of the area; it is usually taken with respect to a gravity axis of the area. An idea of the use of moment of inertia in computing the strength of beams may be obtained by considering the beams supported at both ends and loaded in the middle. The load that the beam will carry is then given by the formula

$P = \frac{8pI}{lh}$, where P is the load in pounds, l the length of the beam in inches, h the height of beam in inches, p the strength of the material, of which the beam is composed, in pounds per square inch, and I is the moment of inertia of the cross section with respect to a horizontal gravity axis. The student should thoroughly master the principles of moment of inertia.

38. Meaning of the Term. — The term *moment of inertia* is somewhat misleading, and the student is apt to try to connect moment of inertia with inertia. The term has no such significance and should be regarded as the name arbitrarily applied to a quantity that engineers frequently use.

Radius of Gyration. The moment of inertia of an area involves area times the square of a distance. We may write $I = \int y^2 dF = Fk^2$, where F is the area and k is a distance, at which, if the area were all concentrated, the moment of inertia would be unchanged. This distance k is called the *radius of gyration.* In a similar way for a mass we write: $I = \int y^2 dM = Mk^2$, and for volume $I = \int y^2 dV = Vk^2$.

39. Units of Moment of Inertia. — The moment of inertia of an area with respect to any axis may be expressed as Fk^2. The area involves square inches, and k is a distance squared. The product is expressed as inches to the fourth power. The moment of inertia of a volume Vk^2 requires inches to the fifth power. The moment of inertia of a mass requires in addition to Vk^2 the factor $\frac{\gamma}{g}$, so that, pounds and feet per second are involved. This is somewhat more complicated since it involves units of weight, distance, and time. This presence of $g(=32.2)$ in the expression requires that all distances be in feet. It is customary to express the moment of inertia of a mass without designating the units used, it being understood that feet, pounds, and seconds were used.

40. Representation of Moment of Inertia. — From the definition of moment of inertia it is evident that an area has a different moment of inertia for every line in its plane. We shall designate the moment of inertia with respect to a line through the center of gravity by I_g with a subscript to indicate the particular gravity line intended. For example, I_{gx} indicates the moment of inertia with respect to a gravity axis parallel to x, and I_{gy} indicates the moment of inertia with respect to a gravity axis parallel to y. The moment of inertia with respect to a line other than a gravity line will be designated by I', the proper subscript indicating the particular line. Similar subscripts will be used to designate the corresponding radii of gyration. It should be noted that moment of inertia is not a quantity involving direction. It has to do only with *magnitude* and is essentially *positive.*

41. Moment of Inertia; Parallel Axes. — Consider the area inclosed by the irregular line (Fig. 67) and suppose that its moment of inertia is known with respect to every line through its center of gravity (gravity axis). It is required to find its moment of inertia with respect to any other line in the plane. Select the arbitrary rectangular axes x and y and draw through the center of gravity of the area a line parallel to the x-axis. Let the distance between these parallel lines be d. The moment of inertia of the area F with respect to the dotted line will be called I_{gx} and its moment of inertia with respect to the x-axis, I'_x. The moment of inertia of the elementary area dF with respect to the x-axis is then written $(d+d')^2 dF$, and the moment of inertia of the whole area F with respect to this same axis becomes

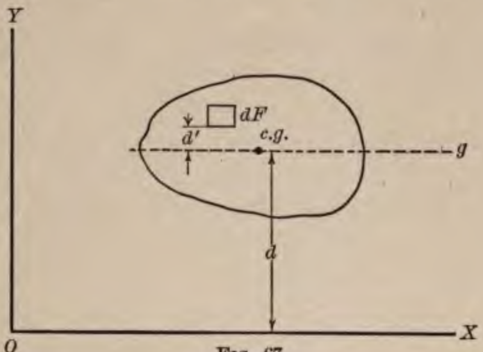


FIG. 67

$$I'_x = \int (d+d')^2 dF = \int d^2 dF + \int 2\,dd'dF + \int (d')^2 dF.$$

But $\int d^2 dF$ may be written $d^2 \int dF = d^2 F$, since d is a constant, and $2\,d \int d'dF = 2\,d(F\bar{d}')$ (see Art. 22), where $\bar{d}'$ is the distance of the center of gravity of the area from the line of reference. In this case $\bar{d}' = 0$, so that $2\,d \int d'dF = 0$. The term $\int (d')^2 dF$ equals I_{gx} by definition (see Art. 37). It follows, then, that

$$\boldsymbol{I'_x = I_{gx} + Fd^2},$$

or, expressed in words,

The moment of inertia of an area with respect to any line in its plane is equal to its moment of inertia with respect to *a parallel gravity axis plus the area times the square of the distance between the two axes.*

This theorem is used very often in work that follows and should be thoroughly understood. It may also be written

$$I_{gx} = I'_x - Fd^2$$

by transposing terms. In this form it is convenient when I'_x, F, and d are known, and I_{gx} is to be determined. It is easily seen from either of these expressions that the moment of inertia of an area for a gravity axis is less than for any other line in the plane.

In case neither axis is a gravity axis, $2\,d\bar{d}'F$ is not equal to zero and $I'_x = I' + Fd^2 + 2\,d\bar{d}'F$, where I' is the moment of inertia with respect to the parallel axis and $\bar{d}'$ the distance of the center of gravity of the area from this parallel axis; the quantities F, d, and I'_x having the same meaning as before.

42. Moment of Inertia; Inclined Axis. — It is often desirable, when I'_x is known, to find the moment of inertia

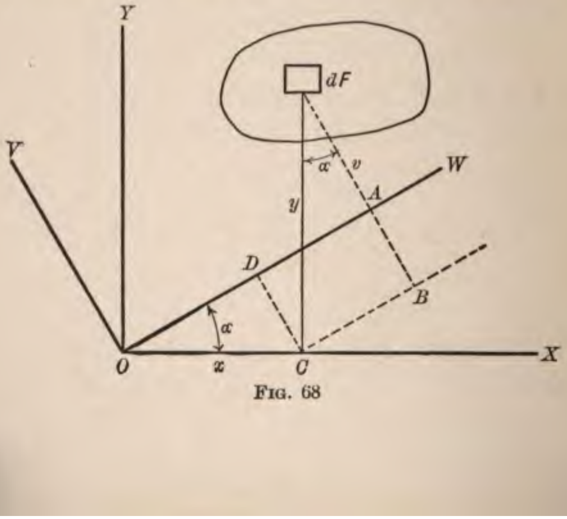


FIG. 68

with respect to an axis w making an angle a with x (see Fig. 68). Here, $I'_w = \int v^2 dF$ and $I'_v = \int w^2 dF$. In terms of x, y, and a,

$$I'_w = \int (y \cos a - x \sin a)^2 dF$$

$$= \int y^2 \cos^2 a dF - 2\int xy \cos a \sin a \, dF + \int x^2 \sin^2 a \, dF$$

$$= \cos^2 a \int y^2 dF - 2 \sin a \cos a \int xy dF + \sin^2 a \int x^2 dF$$

$$= I'_x \cos^2 a - \sin 2a \int xy dF + I'_y \sin^2 a.$$

In a similar way

$$I'_v = I'_x \sin^2 a + 2 \sin a \cos a \int xy dF + I'_y \cos^2 a.$$

These are the required formulæ for obtaining the moment of inertia with respect to inclined axes. It follows that

$$I'_w + I'_v = I'_x + I'_y.$$

That is, the sum of the moments of inertia of an area with respect to two rectangular axes in its plane is the same as the sum of the moments of inertia with respect to any other two rectangular axes in the same plane and passing through the same point. This states that the sum of the moments of inertia for any two rectangular axes through a point is constant. It will be seen in Art. 45 that this constant is the *polar* moment of inertia.

43. Product of Inertia. — The integral $\int xy dF$ is called a *product of inertia*, for want of a better name. In case the area has an axis of symmetry, either the x- or y-axis may be taken along such an axis. The product of inertia then becomes zero, since if x is the axis of symmetry, for every $+y$ there is a corresponding $-y$. A similar reasoning shows the product of inertia zero when y is the axis of symmetry. In such cases

$$I'_w = I'_x \cos^2 a + I'_y \sin^2 a$$

and
$$I'_v = I'_x \sin^2 a + I'_y \cos^2 a.$$

When $\int xydF$ is not equal to zero it is necessary to select the proper limits of integration and sum the integral over the area in question. This is illustrated in Article 53.

44. Axes of Greatest and Least Moment of Inertia. — It is often important to know for what axis through the center of gravity the moment of inertia is least or greatest; that is, what value of a makes I_w or I_v a maximum or a minimum. For any area I_x, I_y, and $\int xydF$ are constant after the x and y axes have been selected. Using the method of the calculus for finding maxima and minima, we have, putting

$$\int xydF = i,\ \frac{dI_w}{da} = (I_y - I_x)\sin 2a - 2i\cos 2a.$$

Equating the right-hand side to zero, the value of a that gives either a maximum or minimum is seen to be given by the equation

$$\tan 2a = \frac{2i}{I_y - I_x},$$

or, what is the same thing,

$$\sin 2a = \frac{2i}{\pm\sqrt{4i^2 + (I_y - I_x)^2}} \text{ and } \cos 2a = \frac{I_y - I_x}{\pm\sqrt{4i^2 + (I_y - I_x)^2}}.$$

It is seen upon substituting these values of $\sin 2a$ and $\cos 2a$ in

$$\frac{d^2 I_w}{da^2} = 2(I_y - I_x)\cos 2a + 4i \sin 2a$$

that the positive sign before the radical indicates a minimum and the negative sign a maximum value for I_w. Investigating the values of a which give I_v maximum or minimum values, it is seen that the value of a for which I_w is a minimum gives I_v maximum, and the value of a for which I_w is a maximum gives I_v minimum. These axes for which the moment of inertia is greatest and least are known as the *Principal Axes of the Area*. This subject will be further discussed in Art. 53.

It is seen from the above that when either the x- or y-axis is a line of symmetry, so that $\int xy\,dF = 0$, the values of a which give maximum or minimum values for I_w and I_v are all zero. This means that the x- and y-axes, themselves, are the principal axes.

45. Polar Moment of Inertia.

45. Polar Moment of Inertia. — The moment of inertia of an area with respect to a line perpendicular to its plane is called the *polar moment of inertia* of the area.

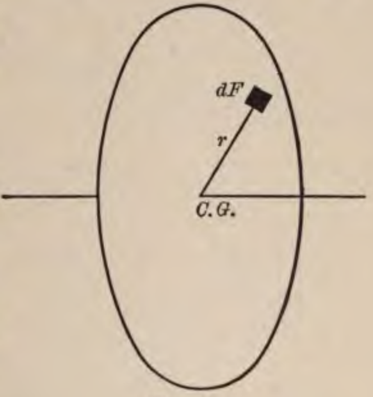


FIG. 69

Consider the area represented in Fig. 69 and let the axis be perpendicular to the area at its center of gravity. Let dF represent an infinitesimal area and let r be its distance from the axis. Representing the polar moment of inertia by I_p, we have

$$I_p = \int r^2 dF;$$

but $$r^2 = x^2 + y^2,$$

so that $$I_p = \int x^2 dF + \int y^2 dF,$$

or $$\boldsymbol{I_p = I_{gy} + I_{gx}}.$$

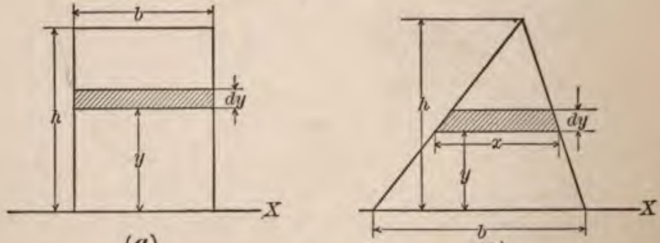
That is, *the polar moment of inertia of an area is equal to the sum of the moments of inertia of any two rectangular axes through the same point.* It has already been shown that $I_x + I_y =$ constant (see Art. 42) for any point of an area.

46. Moment of Inertia of Rectangle. — Let it be required to find the moment of inertia of the rectangle shown in Fig. 70 (a), with respect to the axis, x. We may write

$$I'_x = \int y^2 dF.$$

Since $dF = bdy$, this becomes

$$I'_x = b\int_0^h y^2 dy = \frac{bh^3}{3}.$$



(a) (b)

FIG. 70

To find the moment of inertia with respect to a gravity axis parallel to x it is only necessary to make use of the formula $I_{gx} = I'_x - Fd^2$, from which we have

$$I_{gx} = \frac{1}{12} bh^3 \text{ and } k^2{}_{gx} = \frac{h^2}{12}.$$

From comparison we may write the moment of inertia with respect to a gravity line perpendicular to x,

$$I_{gy} = \frac{1}{12} hb^3 \text{ and } k^2{}_{gy} = \frac{b^2}{12},$$

and the polar moment of inertia for the center of gravity

$$I_p = \frac{bh}{12}(h^2 + b^2).$$

$$k_p{}^2 = \frac{h^2 + b^2}{12}.$$

47. Moment of Inertia of a Triangle. — It is required to find the moment of inertia of the triangle shown in Fig. 70 (*b*) with respect to the axis x, coinciding with the base of the triangle. We have

$I'_x = \int y^2 dF$, where $dF = xdy$,

$$I'_x = \int_0^h y^2 x dy.$$

But $x = \frac{b}{h}(h - y)$, from similar triangles, giving

$$I'_x = \frac{b}{h}\int_0^h y^2(h - y)dy = \frac{1}{12} bh^3 \text{ and } (k'_x)^2 = \frac{h^2}{6}.$$

The moment of inertia with respect to horizontal gravity axis may now be determined. $I_{gx} = I'_x - Fd^2 = \frac{bh^3}{36}$, and $k_{gx}{}^2 = \frac{h^2}{18}$.

It is left as an exercise for the student to find the moment of inertia with respect to an axis through the vertex parallel to the base, and also the polar moment of inertia for the center of gravity.

48. Moment of Inertia of a Circular Area. — The moment of inertia of a circular area with respect to a horizontal gravity axis x, as shown in Fig. 71, may be found as follows: $I_{gx} = \int y^2 dF$. Changing to polar coördinates, remembering that $y = \rho \sin \theta$, and $dF = d\rho(\rho d\theta)$, the integral becomes

$$I_{gx} = \int \rho^2 \sin^2 \theta \rho d\theta d\rho.$$

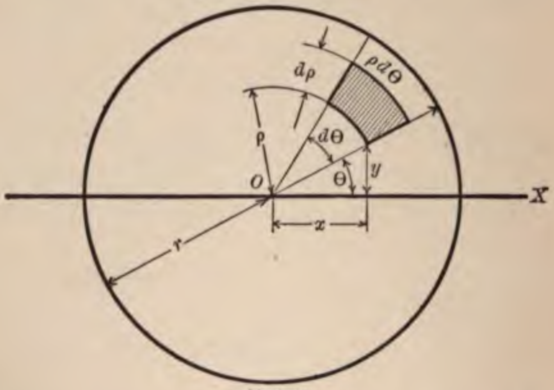


FIG. 71

This integral involves two variables, ρ and θ. It will, therefore, be necessary to make use of a double integration. For this purpose, write

$$I_{gx} = \int_0^{2\pi} \sin^2 \theta d\theta \int_0^r \rho^3 d\rho = \frac{r^4}{4} \int_0^{2\pi} \frac{1}{2}(1 - \cos 2\theta) d\theta$$

$$= \frac{r^4}{4} \left[\frac{\theta}{2} - \frac{1}{4} \sin 2\theta \right]_0^{2\pi} = \frac{\pi r^4}{4}.$$

The corresponding radius of gyration is $k_{gx} = \frac{r}{2}$. On account of the symmetry of the figure this is the moment of inertia for any line in the plane through the center of gravity. It follows that

$$\begin{cases} I_{gy} = \frac{\pi r^4}{4} \\ k_{gy}{}^2 = \frac{r^2}{4} \end{cases} \text{ and that } \begin{cases} I_p = \frac{\pi r^4}{2}. \\ k_p{}^2 = \frac{r^2}{2}. \end{cases}$$

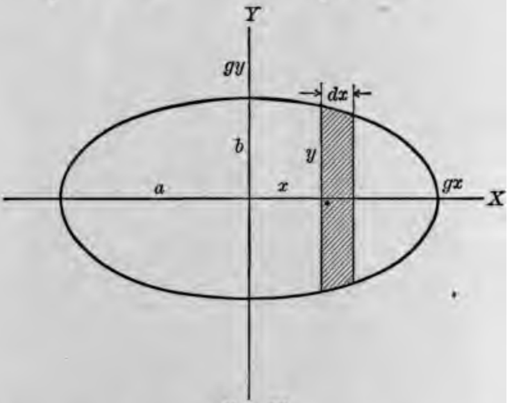


FIG. 72

49. Moment of Inertia of Elliptical Area.—Let it be required to find I_{gx} and I_{gy} of the elliptical area shown in Fig. 72. The equation of the bounding curve is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

and
$$I_{gy} = \int x^2 dF = \int x^2 \, 2\, y dx.$$

From the equation of the bounding curve

$$y = \frac{b}{a}\sqrt{a^2 - x^2},$$

so that

$$I_{gy} = \frac{4\,b}{a}\int_0^{+a} x^2\sqrt{a^2-x^2}\,dx$$

$$= \frac{4\,b}{a}\left[\frac{x}{8}(2\,x^2-a^2)\sqrt{a^2-x^2}+\frac{a^4}{8}\sin^{-1}\frac{x}{a}\right]_0^{+a}$$

$$= \frac{ba^3\pi}{4}, \text{ and therefore } k_{gy} = \frac{a}{2}.$$

In a similar way

$$I_{gx} = \int y^2 dF = \int y^2\, 2\,x dy$$

$$= \frac{4\,a}{b}\int_0^{+b} y^2\sqrt{b^2-y^2}\,dy = \frac{ab^3\pi}{4}, \text{ and therefore } k_{gx} = \frac{b}{2}.$$

Since $I_p = I_{gx} + I_{gy}$, the polar moment of inertia is

$$\frac{ab\pi}{4}(a^2+b^2), \text{ and } k_p = \tfrac{1}{2}\sqrt{a^2+b^2}.$$

It is seen that when $a = b = r$ the equations obtained for the elliptical area are the same as those obtained for the circular area, just as they should be.

50. Moment of Inertia of Angle Section. — When an area may be divided up into a number of triangles, or rectangles, or other simple divisions, the moment of inertia of the whole area with respect to any axis is equal to the sum of the moments of the individual parts. This method is often made use of in determining the moment of inertia of such areas as the section of the angle iron, shown in Fig. 73.

We shall now determine the moment of inertia of this section with respect to the horizontal and vertical gravity axes, I_{gx} and I_{gy}, and also with respect to an axis v (see

Art. 53), making an angle α with the axis x. Consider the section divided into two rectangles, one $5'' \times \frac{5}{8}''$ which we may call F_1 and the other $3\frac{3}{8}'' \times \frac{5}{8}''$, which we may call F_2. The moment of inertia of the section, with respect to x, is equal to the moment of inertia of F_1 with respect to x plus the moment of inertia of F_2 with respect to x, so that $I_{gx} = \frac{1}{12}(5)(\frac{5}{8})^3 + 5(\frac{5}{8})(.808)^2 + \frac{1}{12}(\frac{27}{8})^3\frac{5}{8} + \frac{27}{8}(\frac{5}{8})(1.19)^2$ $= 7.14$ in. to the 4th power. Similarly $I_{gy} = \frac{1}{12}(\frac{27}{8})(\frac{5}{8})^3 + \frac{27}{8}(\frac{5}{8})(1.30)^2 + \frac{1}{12}(\frac{5}{8})(5)^3 + 5(\frac{5}{8})(.88)^2$ $= 12.61$ in. to the 4th power.

NOTE. The problem of finding the moment of inertia of angle sections, channel sections, Z-bar sections, and the built-up sections shown in Figs. 75, 76, 77, 78, 79, is of special interest and importance to engineers, occurring as it does in the computation of the strength of all beams and columns.

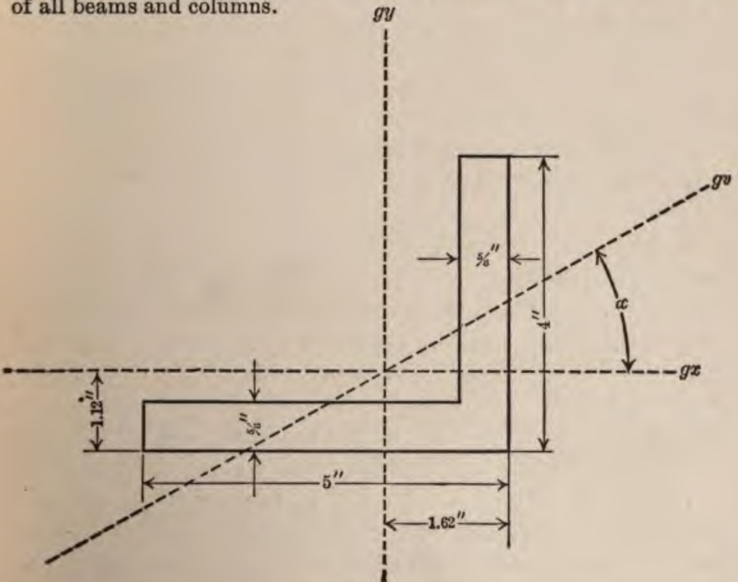


FIG. 73

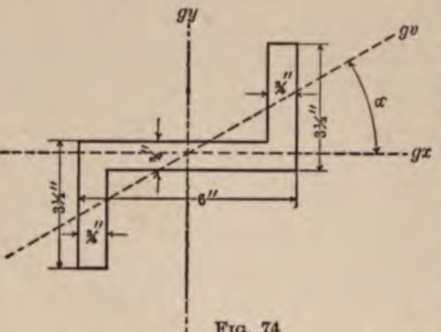


FIG. 74

Problem 62. Find the moment of inertia of the Z-bar section shown in Fig. 74 for the gravity axes g_x and g_y.

HINT. Divide the area into three rectangles.

Problem 63. Compute the moment of inertia for the channel section, shown in Fig. 23, Problem 14, for the horizontal and vertical gravity axes.

Problem 64. Required the moment of inertia of the T-section (Fig. 24, Problem 15), also the moment of inertia of the U-section (Fig. 25, Problem 16) with respect to both horizontal and vertical gravity axes.

Problem 65. The section shown in Fig. 75 consists of a web section and 4 angles, as shown. Find the moment of inertia of the whole section with respect to the horizontal gravity axis. Given, the moment of inertia of an angle section with respect to its own gravity axis, g', is 28.15 in. to the 4th power.

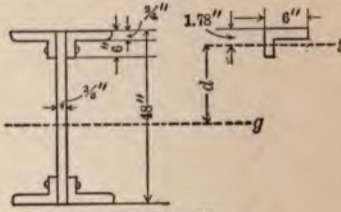


FIG. 75

Problem 66. Consider the section given in Problem 65 to be so taken that it includes two rivet holes, as indicated by the position of the rivets in Fig. 75. Compute the moment of inertia of the whole section, when the moment of inertia of the rivet holes is deducted. The distance from the center of the rivet hole to the outside of the angle section may be taken as 3 in. Compare the result with that obtained in the succeeding problem.

Problem 67. The same section shown in Fig. 75 is shown in Fig. 76 with two cover plates. Find the moment of inertia of the whole beam section with respect to its horizontal gravity axis, now that the cover plates have been added.

Problem 68. Find the moment of inertia of the section of a box girder, shown in Fig. 77, with respect to its horizontal gravity axis. The moment of inertia of one of the angle sections with respect to its own horizontal gravity axis, is 31.92 in. to the 4th power.

Problem 69. Find the moment of inertia of the column section, shown in Fig. 78, with respect to the two gravity axes g_x and g_y. The column is built up of one central plate, two outside plates, and four Z-bars. The legs of the Z-bars are equal, and have a length of 3½ in. The moment of inertia of each Z-bar section with respect to its own horizontal and vertical gravity axis is 42.12 and 15.44 in. to the 4th power, respectively.

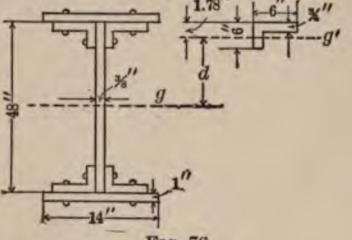


FIG. 76

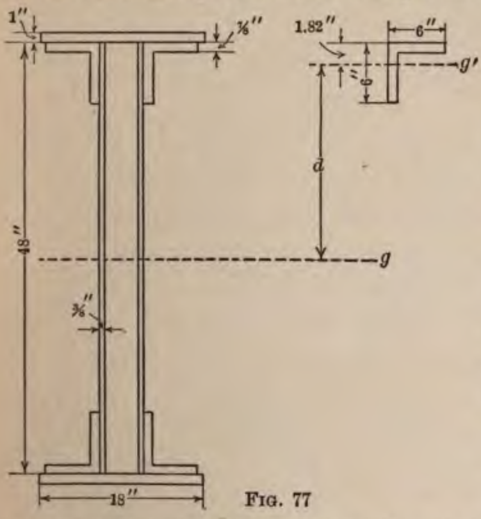


FIG. 77

Problem 70. Find the moment of inertia of the section shown in Fig. 79, with respect to the horizontal and vertical gravity axes g_x and g_y. This section is made up of plates and angles. The moment of inertia of each angle section with respect to both its own horizontal and vertical gravity axes is 28.15 in. to the 4th power.

51. Moment of Inertia by Graphical Method. — It will often be necessary to find the moment of inertia of a plane section whose bounding curve is of a complicated form, as in the case when it is necessary to compute the strength of rails or deck beams. The graphical method given below may be used for such cases.

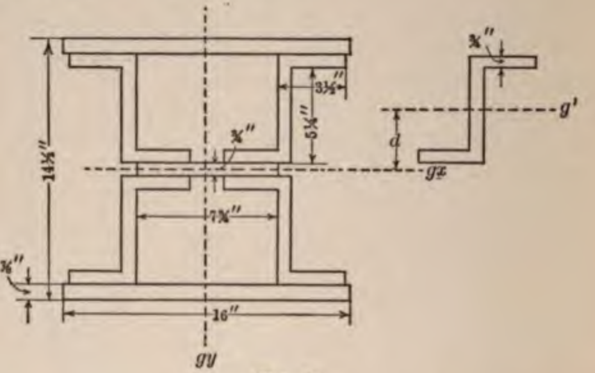


FIG. 78

Let the section be a rail section, Fig. 80. Draw two lines, AB and CD, parallel to the required gravity axis, at any distance, l, apart. The present section is symmetrical with respect to the y-axis, so that it will only be necessary to consider the part on one side of that axis, say the part to the right. Suppose the section divided into strips parallel to AB and CD, and let x denote the length of one of these strips, and dy its width. For each value of x there is a length x' found, such that

$$x' = x\frac{y}{l}.$$

Then for every point P on the original section whose coördinates are x and y, there will be a point P' on the transformed section whose coördinates are x' and y. Suppose all these points, P', constructed, and a boundary line drawn through them. Let F denote the area of the orig-

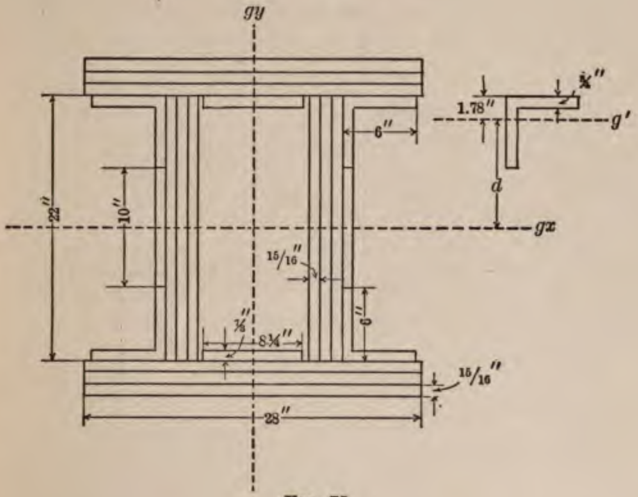


Fig. 79

inal curve, and F' the area of the transformed curve, also let

$$N = \int y dF = \int yx dy = l\int x' dy = lF'.$$

But $\int y dF = \bar{y}F$ (Art. 24), where $\bar{y}$ is the distance of the center of gravity of F from the line AB. It follows that

$$y = l\frac{F'}{F}.$$

This locates the center of gravity.

The moment of inertia will be found by substituting for each x, x', and for each x', x'', such that

$$x'' = x'\frac{y}{l}.$$

Every point, P', now goes over into a point P'', forming a new transformed boundary. Call the area of this last curve F''. Since $x'' = x'\frac{y}{l}$, and $x' = x\frac{y}{l}$, $x'' = x\frac{y^2}{l^2}$.

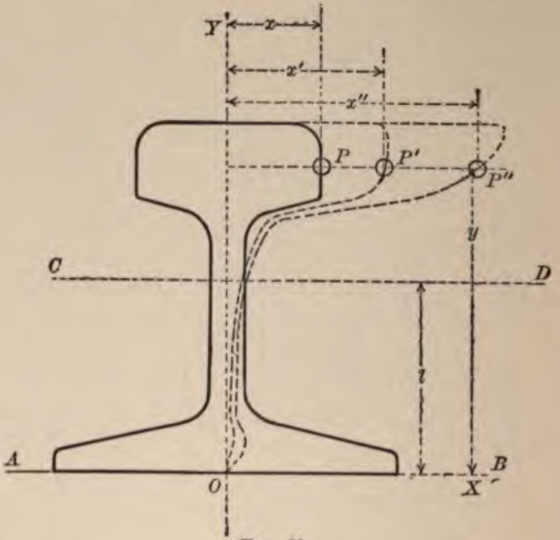


FIG. 80

Therefore the moment of inertia

$$I'_x = \int y^2 dF = \int y^2 x dy = l^2 \int x'' dy = l^2 F'',$$

giving the moment of inertia of the original section with respect to the line *AB*.

To determine I_{gx} it is simply necessary to use the formula $I_{gx} = I'_x - Fd^2$. This gives

$$I_{gx} = l^2\left(F'' - \frac{(F')^2}{F}\right).$$

The areas of the sections are measured by means of a planimeter.

52. Moment of Inertia by Use of Simpson's Rule. — An approximate value for the moment of inertia of irregular sections, such as rail sections, may be obtained by the use of Simpson's Rule. Let the irregular area be the rail section (Fig. 43) and let it be required to find the moment of inertia of the section with respect to the base of the rail. We may write

$$I'_x = \Sigma y^2 A = y_0^2 A_0 + y_1^2 A_1 + y_2^2 A_2 \cdots + y_n^2 A_n,$$

where the A's represent the areas and the y's the distance from the center of the A's to the base of the rail. In this case $y_0 = \frac{1}{4}$, $y_1 = \frac{3}{4}$, $y_2 = \frac{5}{4}$, etc., and $A_0 = 2.95$, $A_1 = 1.95$, $A_2 = .61$, etc. (see Problem 33).

A more exact summation of the terms would be given by adding by means of Simpson's Formula. This gives (Art. 26)

$$I'_x = \frac{6}{3(12)} \Big[y_0^2 u_0 + 4(y_1^2 u_1 + y_3^2 u_3 + y_5^2 u_5 + \cdots) \\ + 2(y_2^2 u_2 + y_4^2 u_4 + \cdots) + y_n^2 u_n \Big],$$

where u_0, u_1, u_2, etc., have the values given in Problem 33.

The student should compare the result obtained by this method with that obtained by the method of direct addition given above. Compare the value obtained with that resulting when Durand's Rule (Art. 28) is used. Use both methods to find the moment of inertia of the sections in Problem 33 and Problem 34.

53. Least Moment of Inertia of Area. — In considering the strength of columns and struts it is necessary to know the least moment of inertia of a cross section, since bending will take place about an axis of its cross section having such least moment of inertia. It was shown in Art. 42 that, if the moment of inertia of the area with respect to two rectangular axes in its plane is known, the moment of inertia with respect to any other axis, making an angle α with one of these, could be found. It was further developed (Art. 44) that the value of α that would render the moment of inertia a minimum was given by the equation

$$\tan 2\alpha = \frac{2\int xydF}{I_y - I_x}.$$

In case either of the axes x or y is an axis symmetry, the value of α given by this criterion is zero, so that, for areas having an axis of symmetry, the axis of least moment of inertia is the axis of symmetry or the one perpendicular to it.

As an illustration of the problem in general let it be required to find the least moment of inertia of the angle section shown in Fig. 73 with respect to any axis in the plane of the area through the center of gravity. Let v be the gravity axis making an angle α with the x-axis. The problem then is to find such a value of α that I_{gv} will be a minimum. From Art. 42 we have

$$I_{gv} = I_{gx}\cos^2\alpha - \sin 2\alpha\int xydxdy + I_{gy}\sin^2\alpha.$$

In Art. 50 it was found that $I_{gx} = 7.14$ and $I_{gy} = 12.61$. We proceed now to find the value of $\int xydxdy$ for the angle section. For this purpose, suppose the section composed of two rectangles, F_1 (5 in. × $\frac{5}{8}$ in.), and F_2 ($\frac{27}{8}$ in. × $\frac{5}{8}$ in.), and then find the value of the integral, for the two rectangles separately. Considering first the area F_1, and using the double integration, we get

$$\int_{1.62}^{-3.38}\int_{-.495}^{-1.12} xydxdy = \int_{1.62}^{-3.38} xdx\left[\frac{(-1.12)^2}{2} - \frac{(-.495)^2}{2}\right]$$

$$= .505\left[\frac{(-3.38)^2}{2} - \frac{(1.62)^2}{2}\right] = 2.222.$$

In a similar way for F_2, we have

$$\int_{.995}^{1.62}\int_{-.495}^{2.88} xy\,dx\,dy = \int_{.995}^{1.62} x\,dx\left[\frac{(2.88)^2}{2} - \frac{(-.495)^2}{2}\right]$$

$$= 4.025\left[\frac{(1.62)^2}{2} - \frac{(.995)^2}{2}\right] = 3.288.$$

Therefore, $\int xy\,dx\,dy$ for the whole area of the angle section is 5.51 in. to the 4th power. From this we find

$$\tan 2\,a = \frac{11.02}{5.47} = 2.02.$$

Therefore $\qquad 2\,a = 63°40',$

$$a = 31°\ 50'.$$

The expression for I_{gv} now becomes

$$I_{gv} = 7.14\cos^2(31°\ 50') - 5.51\sin(63°\ 45')$$
$$+ 12.61\sin^2(31°\ 50') = 3.72 \text{ in. to the 4th power.}$$

This gives the least radius of gyration,

$$k_{gv} = .84 \text{ in.}$$

Problem 71. Find the least moment of inertia I_{gv} and least radius of gyration k_{gv} of the Z-section shown in Fig. 74. In this case $I_{gx} = 15.44$ in. to the 4th power and $I_{gy} = 42.12$ in. to the 4th power.

Ans. Least $I_{gv} = 5.66$ in. to 4th power and least $k_{gv} = .81$ in.

Problem 72. An angle iron has equal legs. The section, similar to that in Fig. 73, is 8 in. × 8 in. with a thickness of $\frac{1}{2}$ in. Find I_{gx}, I_{gy}, least I_{gv} and least k_{gv}.

Ans. $I_{gx} = I_{gy} = 48.65$ in. to the 4th power,
$I_{gv} = 19.59$ in. to the 4th power,
$k_{gv} = 1.59$ in.

Problem 73. Find the moment of inertia of column section, shown in Fig. 78, with respect to an axis v making an angle of 30° with gx. What value of α gives I_{gv} minimum in this case?

54. The Ellipse of Inertia. — It is interesting to note, at this point, the relations between the moments of inertia with respect to all the lines, in the plane of the area passing through a point. We have seen that for every point in an area there is always a pair of rectangular axes for which the moment of inertia is a maximum or a minimum; that is, there is always a pair of principal axes. The criterion for such axes was found to be

$$\tan 2\alpha = \frac{2(i)}{I_y - I_x},$$

which means, since the tangent of an angle may have any value from 0 to infinity, positive and negative, that for every point there is always a pair of axes such that

$$i = 0, \text{ or } \int xy\,dF = 0.$$

This means that the expression for I_v may always be reduced to the form

$$I_v = I_x \cos^2\alpha + I_y \sin^2\alpha$$

by properly selecting the axes of reference, where now I_x and I_y represent the principal moments of inertia. If we divide through by F, the equation becomes

$$k_v^2 = k_x^2 \cos^2\alpha + k_y^2 \sin^2\alpha.$$

Let $\rho = \frac{k_y k_x}{k_v}$, so that $k_y = \frac{k_v \rho}{k_x}$ and $k_x = \frac{k_v \rho}{k_y}$,

then $k_v^2 = \frac{\rho^2 k_v^2}{k_y^2} \cos^2\alpha + \frac{\rho^2 k_v^2}{k_x^2} \sin^2\alpha,$

or dividing by k_v^2,

$$1 = \frac{\rho^2 \cos^2\alpha}{k_y^2} + \frac{\rho^2 \sin^2\alpha}{k_x^2},$$

which is the equation of an ellipse referred to the principal axes of inertia as axes. It may be written

$$\frac{x^2}{k_y^2}+\frac{y^2}{k_x^2}=1.$$

It is evident that k_v is inversely proportional to ρ, so that the major axis of the ellipse is along the axis of the least moment of inertia and minor axis along the axis of greatest moment of inertia. The ellipse of inertia has no physical significance, but merely shows the relation between the moments of inertia with respect to the different axes through a point. If, then, the moments of inertia for all axes in a plane, through a point, be laid off on these axes, to scale, the locus of the end points will be an ellipse. The ellipse of inertia furnishes a graphical method for finding the moment of inertia for any axis through a point.

55. Moment of Inertia of Thin Plates. — Suppose the plate of constant thickness t and unit weight γ and let x be the distance of any dM from the axis of reference, then $I_x=\int x^2dM=\frac{\gamma}{g}\int x^2dV=\frac{\gamma}{g}\,t\int x^2dF.$ But this expression under the integral sign is the expression for the moment of inertia of the area of one of the faces of the plate, if F represents the area of a face. Therefore, *the moment of inertia of a thin plate with reference to an axis in its plane equals* $\frac{\gamma t}{g}$ *times the moment of inertia of the area of its face with reference to the same axis.*

A similar statement is seen to hold with reference to the polar moment of inertia of a thin plate by replacing x by ρ the distance of dM from a point. The following results are deduced at once.

I, for thin circular plate:

$$I_{gx}=\frac{\gamma t}{g}\left(\frac{\pi r^4}{4}\right)=M\frac{r^2}{4}\,;\ k_{gx}=\frac{r}{2}\,;$$

$$I_{pg}=\frac{\gamma t}{g}\left(\frac{\pi r^4}{2}\right)=\frac{Mr^2}{2}\,;\ k_{pg}=\frac{r}{\sqrt{2}}.$$

I, for thin rectangular plate:

$$I_{gx}=\frac{\gamma t}{g}\left(\frac{1}{12}bh^3\right)=\frac{Mh^2}{12}\,;\ k_{gx}=\frac{h}{2\sqrt{3}}\,;$$

$$I_{pg}=\frac{\gamma t}{g}\left(\frac{1}{12}bh^3+\frac{1}{12}b^3h\right)=\frac{M}{12}\left(h^2+b^2\right);$$

$$k_{pg}=\frac{\sqrt{h^2+b^2}}{2\sqrt{3}}$$

I, for triangular plate:

I_{gx} (parallel to base) $=\frac{\gamma t}{g}\left(\frac{1}{36}bh^3\right)=\frac{Mh^2}{18}\,;\ k_{gx}=\frac{h}{3\sqrt{2}}.$

I, for elliptical plate:

(1) major axis:

$$I_{gx}=\frac{\gamma t}{g}\left(\frac{\pi ab^3}{4}\right)=\frac{Mb^2}{4}\,;\ k_{gx}=\frac{b}{2}.$$

(2) minor axis:

$$I_{gy}=\frac{\gamma t}{g}\left(\frac{\pi ba^3}{4}\right)=\frac{Ma^2}{4}\,;\ k_{gy}=\frac{a}{2}.$$

(3) polar axis:

$$I_{gp}=\frac{\gamma t}{g}\,\frac{\pi ab}{4}(b^2+a^2)=\frac{M(b^2+a^2)}{4}\,;\ k_{pg}=\frac{\sqrt{b^2+a^2}}{2}$$

(4) central axis, $2r$, making angle α with major axis:

$$I_{gv}=\frac{\gamma t}{g}\left(\frac{\pi a^3b^3}{4r^2}\right)=\frac{Ma^2b^2}{4r^2}\,;\ k_{gv}=\frac{ab}{2r}.$$

56. Moment of Inertia of Right Prism; Geometrical Axis. — The moment of inertia of a right prism of height h with respect to its geometrical axis is given by the expression

$$I_{gy} = \int \rho^2 dM = \frac{\gamma h}{g} \int \rho^2 dF.$$

That is, *the moment of inertia of a right prism of height h is equal to $\frac{\gamma h}{g}$ times the polar moment of inertia of its base.*

From this result we may write:

For right circular cylinder:

$$I_{gy} = \frac{\gamma h}{g}\left(\frac{\pi r^4}{2}\right) = \frac{Mr^2}{2}\ ;\ k_{gy} = \frac{r}{\sqrt{2}}.$$

Right parallelopiped:

$$I_{gy} = \frac{\gamma h}{g}\left(\frac{1}{12}h_1 b^3 + \frac{1}{12} b h_1^3\right) = \frac{M}{12}(b^2 + h_1^2),\ k_{gy} = \frac{\sqrt{b^2 + h_1^2}}{2\sqrt{3}};$$

therefore,

$$I_{gy} = \frac{Md^2}{12}\ ;\ k_{gy} = \frac{d}{2\sqrt{3}},$$

where d is the diagonal of the base.

Hollow cylinder:

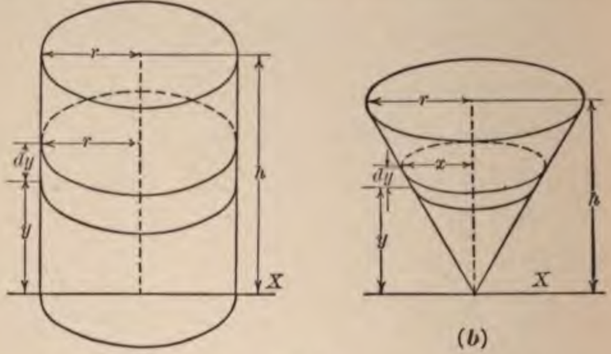
$$I_{gy} = \frac{\gamma h}{g}\left(\frac{\pi r_1^4}{2} - \frac{\pi r_2^4}{2}\right) = \frac{M}{2}(r_1^2 + r_2^2)\ ;\ k_{gy} = \frac{\sqrt{r_1^2 + r_2^2}}{\sqrt{2}}.$$

Elliptical cylinder:

$$I_{gy} = \frac{\gamma h}{g}\frac{\pi ab}{4}(b^2 + a^2) = \frac{M(b^2 + a^2)}{4}\ ;\ k_{gy} = \frac{\sqrt{b^2 + a^2}}{2}.$$

57. Moment of Inertia of Right Prism; Axis Perpendicular to Geometrical Axis. — Let the axis be perpendicular to the geometrical axis through the base. Consider a thin slice cut from the prism by two parallel planes, distant dy

and perpendicular to the prism. The slice so cut may be considered a thin plate, whose mass is $\frac{\gamma}{g}dyF$, where F is the cross section of the prism. Suppose the distance of this slice from the base is y. Then the moment of inertia of this slice with respect to the axis through the base is equal to its moment of inertia about its own parallel gravity axis plus its mass times the square of the distance between the axes. (See Art. 41.) This will be made more clear by reference to the special case of the right circular cylinder of Fig. 81 (a). Adding the moments of all the slices, we have



(a) (b) FIG. 81

$$I'_x = \int_0^h \left(\frac{\gamma}{g} F dy \cdot k_p^2 + \frac{\gamma}{g} F dy \cdot y^2\right)$$

$$= \frac{\gamma h}{g} F k_p^2 + \frac{\gamma F h^3}{3g} = M\left(k_p^2 + \frac{h^2}{3}\right).$$

For the right circular cylinder:

$$I_x' = M\left(\frac{r^2}{4} + \frac{h^2}{3}\right), \quad k_x' = \sqrt{\frac{r^2}{4} + \frac{h^2}{3}}.$$

For right elliptical cylinder:

(1) major axis: $I'_x = M\left(\frac{b^2}{4}+\frac{h^2}{3}\right).$

(2) minor axis: $I'_x = M\left(\frac{a^2}{4}+\frac{h^2}{3}\right).$

For right rectangular cylinder:

(1) parallel to b: $I'_x = M\left(\frac{h_1^2}{12}+\frac{h^2}{3}\right).$

(2) parallel to h_1: $I'_x = M\left(\frac{b^2}{12}+\frac{h^2}{3}\right).$

58. Moment of Inertia of Solid of Revolution. — Consider the moment of inertia of a solid of revolution with respect to its axis of revolution. Imagine the solid cut into thin slices, all of same thickness, by parallel planes perpendicular to the axis of revolution. Each slice is a circular disk of thickness dy and radius x, and its polar moment of inertia with respect to the axis of revolution is $\frac{\gamma}{g}dy\pi x^2 \cdot \frac{x^2}{2}.$ The moment of inertia of the solid of revolution is the sum of the moments of the small slices, so that

$$I_{gr} = \int \frac{\gamma\pi}{2g}x^4 dy,$$

the limits of integration and the relation between x and y depending upon the particular solid considered.

For right circular cone:

The right circular cone is illustrated in Fig. 81 (b). For this case

$$I_{gr} = \int_0^h \frac{\gamma\pi}{2g}x^4 dy = \frac{\gamma\pi}{2g}\frac{r^4}{h^4}\int_0^h y^4 dy = \frac{\gamma\pi r^4 h}{10g} = \frac{3}{10}Mr^2,$$

since $x = \frac{r}{h}y.$

For a sphere:

$$I_{gr} = \int_{-r}^{+r} \frac{\gamma\pi}{2g} x^4 dy = \frac{\gamma\pi}{2g}\int_{-r}^{+r}(r^2 - y^2)^2 dy,$$

since $$x^2 = r^2 - y^2$$

$$= \frac{\gamma\pi}{2g}\int_{-r}^{+r}(r^4 dy - 2r^2y^2dy + y^4dy) = \left[r^4y - \frac{2r^2y^3}{3} + \frac{y^5}{3}\right]_{-r}^{+r}$$

$$= \frac{\gamma\pi}{g} \cdot \frac{8}{15} r^5 = \frac{2}{5} \boldsymbol{M r^2} \text{ therefore } k_{gr}^2 = \frac{2}{5} r^2.$$

For an ellipsoid of revolution:

(1) prolate spheroid:

$$I_{gr} = \frac{\gamma\pi}{2g}\int_{-a}^{+a} y^4 dx = \frac{\gamma\pi}{2g}\frac{b^4}{a^4}\int_{-a}^{+a}(a^2 - x^2)^2 dx,$$

since $$y = \frac{b}{a}\sqrt{a^2 - x^2},$$

and $$v = \frac{4}{3}\pi ab^2,\ I_{gr} = \frac{\gamma\pi b^4}{ga^4} \cdot \frac{8}{15} a^5 = \frac{2}{5}\boldsymbol{M b^2}.$$

(2) oblate spheroid:

$$I_{gr} = \frac{\gamma\pi}{2g}\int_{-b}^{+b} x^4 dy = \frac{\gamma\pi a^4}{2gb^4}\int_{-b}^{+b}(b^2 - y^2)^2 dy$$

$$= \frac{\gamma\pi a^4}{gb^4} \cdot \frac{8}{15} b^5 = \frac{2}{5}\boldsymbol{M a^2},$$

since $$x = \frac{a}{b}\sqrt{b^2 - y^2}, \text{ and } v = \frac{4}{3}\pi ba^2.$$

59. Moment of Inertia of Right Circular Cone. — When the moment of inertia of a right circular cone with respect to an axis through its vertex parallel to the base is to be found we may proceed as in Art. 58.

Imagine the cone to be cut by parallel planes into slices as shown in Fig. 81 (*b*). The moment of inertia of the whole cone with respect to x is equal to the sum of the moments of the small slices with respect to x.

Then $$I'_x = \int_0^h\left(\frac{\gamma\pi}{g}\frac{x^4}{4}dy + \frac{\gamma\pi x^2 dy}{g}y^2\right)$$

$$= \frac{\gamma\pi}{g}\int_0^h\left(\frac{x^4}{4}dy + x^2y^2dy\right);$$

but $$x = \frac{r}{h}y,$$

so that $$I'_x = \frac{\gamma\pi}{4g}\frac{r^4h}{5} + \frac{\gamma\pi}{g}\frac{r^2}{5}h^3 = M\left(\frac{3}{20}r^2 + \frac{3}{5}h^2\right).$$

60. Moment of Inertia of Mass; Parallel Axis. — If the moment of inertia of a mass with respect to an axis in space is known, it is important to be able to determine its moment with respect to any parallel axis. Let dM be the mass of a particle of the body, a' and a (Fig. 82) the parallel axes distant d, and r' and r the distances of the mass from the two axes.

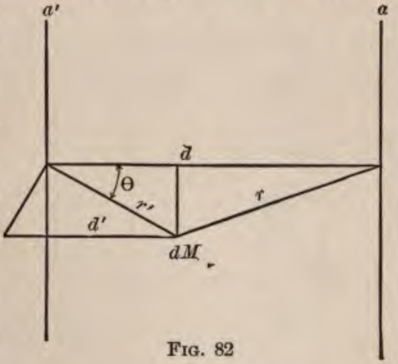


FIG. 82

From the triangle $r^2 = r'^2 + d^2 - 2\,r'd\cos\theta$, multiplying by dM and integrating over the body, we have

$$\int r^2 dM = \int r'^2 dM + \int d^2 dM - \int 2\,r'd\cos\theta\, dM.$$

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But $r' \cos\theta = d'$, the distance of dM from a plane through a', perpendicular to the plane a' and a,

so that
$$\int r^2 dM = \int r'^2 dM + d^2 M - 2\,d \int d' dM,$$

or
$$I_a = I_{a'} + d^2 M - 2\,dM\bar{d}',$$

where $\bar{d}'$ represents the distance of the center of gravity of the mass from the plane through a' perpendicular to the plane of a' and a.

From this relation, if the position of the body is known and its moment of inertia with respect to an axis in space, its moment of inertia with respect to any other parallel axis may be found.

In particular, if $\bar{d}' = 0$, that is, if the center of gravity of the body lies in a plane through a' perpendicular to the plane of a' and a, the relation reduces to

$$I_a = I_{a'} + Md^2.$$

That is, *the moment of inertia of a mass with respect to any axis in space is equal to its moment of inertia with respect to a parallel axis, lying in a gravity plane, perpendicular to the line joining the two axes, plus the mass times the square of the distance between the bodies.*

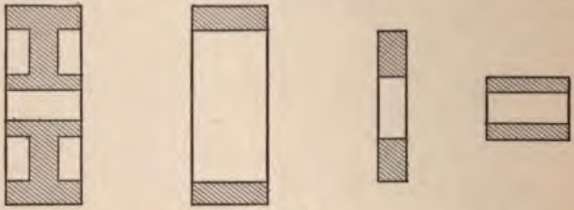


FIG. 83

The student will notice that this relation is very similar to the one developed for the moment of inertia of plane areas with respect to parallel axes, in Art. 41.

Problem 74. Show that the moment of inertia of a right circular cylinder, altitude h, and radius of base r, with respect to a gravity axis parallel to the base is $I_{gx} = M\left(\frac{r^4}{4}+\frac{h^2}{12}\right)$, and find the moment of inertia with respect to an axis parallel to this and at a distance d from the base.

Problem 75. Show that the moment of inertia of a right circular cone, altitude h, and radius of base r, with respect to a gravity axis parallel to the base is $I_{gx} = \frac{3}{20} M\ (r^2 + \frac{1}{2} h^2)$.

Problem 76. Find the moment of inertia of a slender rod of length l with respect to an axis through one end and perpendicular to the rod. Let the cross section be F and the mass M; then $I = {}^1 M l^2$.

Problem 77. It is required to find the moment of inertia of the cast-iron disk fly wheel shown in Fig. 83 with respect to its geometrical axis.

HINT. The wheel may be regarded as made up of three hollow cylinders, the moment of the whole wheel being equal to the sum of the moments of the three parts. The dimensions are as follows: diameter of wheel 2 ft., width of rim and hub 4 in., thickness of rim and web 2 in., thickness of hub $1\frac{1}{4}$ in., and diameter of shaft 2 in. All distances must be in feet.

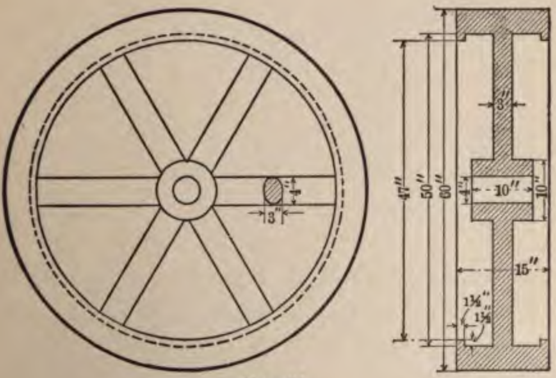


FIG. 84

Problem 78. Find the moment of inertia of the cast-iron fly wheel shown in Fig. 84 with respect to its axis of rotation. There are six elliptical spokes, and these may be regarded as of the same cross section throughout their entire length.

61. Moment of Inertia of Non-homogeneous Bodies.—When the bodies are not homogeneous, the expressions for the moment of inertia given in this chapter do not hold, since in that case γ is no longer constant. In case the law of variation of γ is known, as for example if γ varies as the distance from the line, then this variable value of γ may be used and the moment of inertia found.

Let it be required to find the moment of inertia of a right circular cylinder with respect to an axis through its base, if the density varies as the distance from the base, in such a way that $\gamma = y$, where y is a distance measured from the base (Art. 57).

Then
$$I'_x = \int_0^h \left(\frac{\gamma}{g} F dy\, k_p^2 + \frac{\gamma}{g} F dy y^2\right)$$

$$= \left[\frac{k_p^2 F}{g} \cdot \frac{y^2}{2} + \frac{F}{g} \cdot \frac{y^4}{4}\right]_0^h = \frac{Fh}{g}\left(k_p^2 \frac{h}{2} + \frac{h^3}{4}\right).$$

If γ varies in some other way, the proper value must be used in the integral. In most cases, however, γ is a constant.

62. Moment of Inertia of a Mass; Inclined Axis.—We shall now study the problem of finding the moment of inertia of a solid with respect to an axis inclined to the coördinate axes. Suppose the moments of inertia of the body with respect to the three coördinate axes known from the expressions:

$$I_x = \int (z^2 + y^2)\, dM,$$

$$I_y = \int (z^2 + x^2)\, dM,$$

$$I_z = \int (y^2 + x^2)\, dM,$$

and let it be required to find the moment of inertia of the body with respect to any other axis OA making angles α, β, γ with the coördinate axes. (See Fig. 85.) Let dM equal the mass of an infinitesimal portion of the body and d its distance from the axis OA.

Since $r^2 = x^2 + y^2 + z^2$, $OA = x \cos\alpha + y \cos\beta + z \cos\gamma$ and

$$d^2 = r^2 - \overline{OA}^2 = (x^2 + y^2 + z^2) - (x \cos\alpha + y \cos\beta + z \cos\gamma)^2,$$

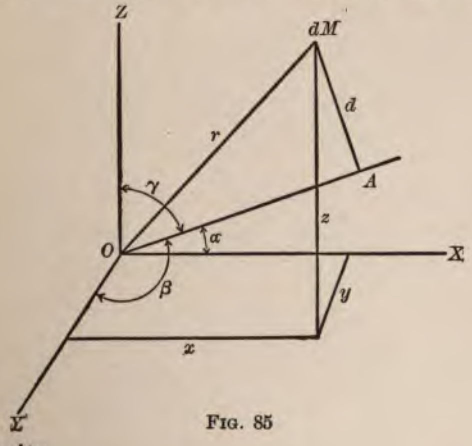


FIG. 85

we may write

$$I_{OA} = \int d^2 dM$$

$$= \int [(x^2 + y^2 + z^2) - (x \cos\alpha + y \cos\beta + z \cos\gamma)^2]\, dM.$$

This reduces, since $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$, to

$$\begin{aligned}I_{OA}=\int(y^2+z^2)\cos^2\alpha dM+\int(z^2+x^2)\cos^2\beta dM\\+\int(x^2+y^2)\cos^2\gamma dM-2\cos\alpha\cos\beta\int xydM\\-2\cos\beta\cos\gamma\int yzdM-2\cos\gamma\cos\alpha\int xzdM,\end{aligned}$$

or

$$\begin{aligned}I_{OA}=I_x\cos^2\alpha+I_y\cos^2\beta+I_z\cos^2\gamma-2\cos\alpha\cos\beta\int xydM\\-2\cos\beta\cos\gamma\int yzdM-2\cos\gamma\cos\alpha\int xzdM,\end{aligned}$$

which gives the moment of inertia of the body with respect to an inclined axis in terms of the moments of inertia with respect to the coördinate axes and the products of inertia $\int xydM$, $\int yzdM$, and $\int xzdM$.

63. Principal Axes. — If the three products of inertia $\int xydM$, $\int yzdM$, and $\int xzdM$ are each equal to zero, the expression for I_{OA} reduces to the form

$$I_{OA}=I_x\cos^2\alpha+I_y\cos^2\beta+I_z\cos^2\gamma.$$

In this case the coördinate axes x, y, and z are called the *principal axes* for the point O and the moments I_x, I_y, and I_z the *principal moments of inertia.*

If the point O is the center of gravity of the body and the products of inertia are each equal to zero, the principal axes are called the *principal axes* of the body. It can be shown that it is always possible to select the coördinate axes x, y, and z so that the products of inertia given in the expression for I_{OA} will each be zero. It follows that for every point of a body there exists a set of rectangular axes that are principal axes.

Problem 79. Find the moment of inertia of the ellipsoid whose surface is given by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

with respect to the axes a, b, and c, and with respect to an inclined axis making angles α, β, γ with a, b, and c, respectively. The volume of an ellipsoid is

$$\frac{4\,\pi abc}{3},\ I_a=\frac{M}{5}(b^2+c^2),\ I_b=\frac{M}{5}(c^2+a^2),\ I_c=\frac{M}{5}(a^2+b^2)$$

and $$I_{OA}=I_a\cos^2\alpha+I_b\cos^2\beta+I_c\cos^2\gamma.$$

64. Ellipsoid of Inertia. — It is always possible to reduce the expression for I_{OA} to the form

$$I_{OA}=I_x\cos^2\alpha+I_y\cos^2\beta+I_z\cos^2\gamma,$$

by selecting the axes x, y, and z so that the products of inertia are zero.

Dividing this equation through by M, we have

$$k_{OA}^2=k_x^2\cos^2\alpha+k_y^2\cos^2\beta+k_z^2\cos^2\gamma.$$

Let $\rho=\dfrac{k_xk_yk_z}{k_{OA}}$, so that

$$k_x=\frac{\rho k_{OA}}{k_yk_z},\ k_y=\frac{\rho k_{OA}}{k_xk_z},\ \text{and } k_z=\frac{\rho k_{OA}}{k_xk_y},$$

so that

$$k_{OA}^2=\frac{\rho^2k_{OA}^2\cos^2\alpha}{k_z^2k_y^2}+\frac{\rho^2k_{OA}^2\cos^2\beta}{k_x^2k_z^2}+\frac{\rho^2k_{OA}^2\cos^2\gamma}{k_x^2k_y^2},$$

which when divided through by k_{OA}^2 becomes

$$1=\frac{(\rho\cos\alpha)^2}{k_z^2k_y^2}+\frac{(\rho\cos\beta)^2}{k_x^2k_z^2}+\frac{(\rho\cos\gamma)^2}{k_x^2k_y^2}.$$

This is seen to be the equation of an ellipsoid whose semi-axes are $k_z k_y$, $k_x k_z$, and $k_x k_y$; the equation may be written

$$1 = \frac{x'^2}{k_z^2 k_y^2} + \frac{y'^2}{k_x^2 k_z^2} + \frac{z'^2}{k_x^2 k_y^2},$$

where x', y', and z' represent the coördinates of a point on the line OA at a distance ρ from O.

It is evident that if we draw all the lines through O and then locate all points x', y', and z' on these lines, such that $\rho = \frac{\text{constant}}{k}$, the locus of all the points will be the ellipsoid of inertia for that point. For the position of the coördinate axes selected, the principal axes of the ellipsoid of inertia coincide with the principal axes of the body.

Problem 80. Write the equation and construct the inertia ellipsoid for the center of gravity of a right circular cylinder, altitude h and radius r.

Problem 81. Construct the inertia ellipsoid for the center of a solid sphere of radius r.

Problem 82. Show that the moment of inertia of the segment of the circle F_1 (Fig. 86) with respect to the axis OZ is

$$\frac{r^4}{4}\left(\frac{\alpha}{2} + \frac{1}{2}\sin\alpha\right),$$

the moment of the sector $OBSA$, minus $\frac{1}{3}\ ah^3$, the moment of the triangle OAB or

$$I_{OZ} = \frac{r^4}{16}\left(2\,\alpha - \sin 2\,\alpha\right),$$

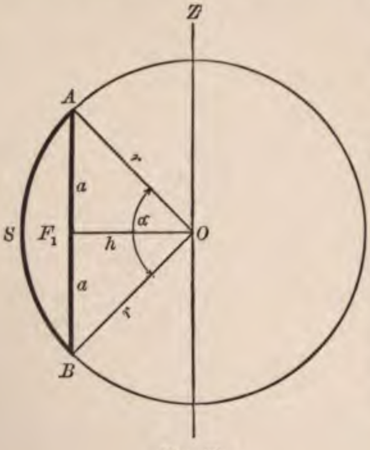


FIG. 86

and the moment of inertia of F_1 with respect to OS is $\frac{r^4}{4}\left(\frac{\,}{2}-\frac{1}{2}\sin\alpha\right)$, the moment of inertia of the sector, minus $\frac{1}{6}ha^3$, the moment of inertia of the triangle AOB or $I_{OS}=\frac{r^4}{8}\left[\alpha-\sin\alpha\left(1+\frac{2}{3}\sin^2\frac{\alpha}{2}\right)\right]$.

Problem 83. Show that the moment of inertia of the counterbalance, Fig. 37, with respect to a line through O, perpendicular to OS, and in the plane of the wheel, is

$$I_{OZ}=\left[\frac{r^4}{16}\left(2\alpha-\sin 2\alpha\right)-\frac{r_1{}^4}{16}\left(2\beta-\sin 2\beta\right)+\frac{4\,a^3(OO')}{3}-F_2(OO')^2,\right]\frac{\gamma t}{g}$$

where $F_2=\frac{\beta r_1^2}{2}-ar_1\cos\frac{\beta}{2}$, $OO'=r_1\cos\frac{\beta}{2}-r\cos\frac{\alpha}{2}$, and t is the thickness, as explained in Art. 25.

Problem 84. Find the moment of inertia of the counterbalance, Fig. 37, with respect to a line through O perpendicular to the plane of the wheel. It may be written

$$I_0=\left\{\frac{r^4}{16}\left[4\,\alpha-\sin 2\,\alpha-2\sin\alpha\left(1+\frac{2}{3}\sin^2\frac{\alpha}{2}\right)\right]-\frac{r_1{}^4}{16}\left[4\,\beta-\sin 2\,\beta-\right.\right.$$

$$\left.\left.2\sin\beta\left(1+\frac{2}{3}\sin^2\frac{\beta}{2}\right)\right]+\frac{4\,a^3(OO')}{3}-F_2(OO')^2\right\}\frac{\gamma t}{g}.$$

65. Moment of Inertia of Locomotive Drive Wheel. — The drive wheel may be represented as in Fig. 87, and may be considered as made up of a tire, rim, twenty elliptical spokes, counterbalance, and equivalent weight on opposite side of center, and hub. The tire, rim, and hub may each be considered as hollow cylinders whose moments may be found as in Art. 56. The moment of the spokes is easily found by considering them elliptic cylinders (see Art. 57), with the short axis of the ellipse in the plane of the wheel. The moment of the counterbalance is equal to the moment of the weights carried by the crank pin, times $\frac{\text{radius of counterbalance}}{\text{radius of crank}}$.

The dimensions of the wheel are as follows: radius of tread 40″, radius of inside of tire 36″, width of tread 5″, outside radius of rim 36″, inside radius of rim 34″, width of rim $4\frac{1}{8}''$. There are twenty elliptical spokes 24″ long, $3\frac{1}{4}'' \times 2\frac{1}{4}''$. The hub is 10″ outside radius, $4\frac{3}{4}''$ inside radius, and 8″ thick, radius of crank pin circle 18″. The counterbalance has an outside radius of 34″, an inside radius of 7′ 11.5″, thickness $7\frac{1}{4}''$, mass 20.2, and distance of its center of gravity from the center 28.8″, $\alpha = 94° 40'$, $\beta = 30° 20'$, $\gamma = 490$ (Art. 64).

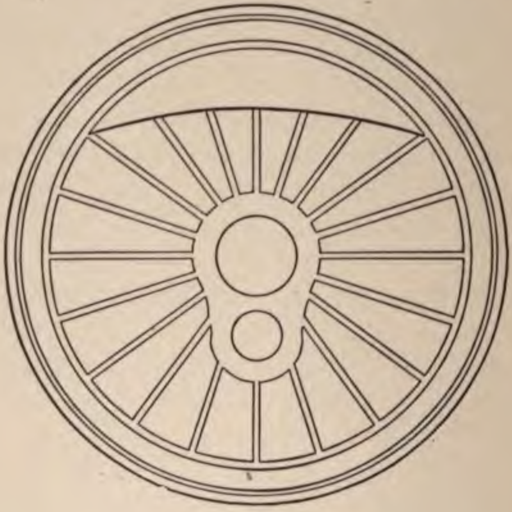


FIG. 87

We shall neglect the moment of inertia of the flange and shall consider the spokes cylindrical throughout their length, and that 10% of the moment of inertia of the spokes is included in that of the counterbalance and boss. The moment of inertia of the wheel with respect to its axis of rotation will first be found. The moment of inertia consists of: I_0 for tire $= 415$, I_0 for rim $= 139$, I_0 for spokes $= 246$, I_0 for hub $= 7$, I_0 for counterbalance $= 344$, and for boss $= 73$. The total moment is 1224.

With respect to a gravity axis OZ, Fig. 86, in the plane of the wheel when the counterbalance is in a position where the line joining its center of gravity to the center is perpendicular to OZ, we get for the moment of the various parts: I_{oz} for tire $= 207$, I_{oz} for rim $= 69$, I_{oz} for spokes $= 123$, I_{oz} for hub $= 3$, I_{oz} for counterbalance $= 279$, and for boss $= 73$. The total moment is 755. In computing the moment of the spokes in this case, it was necessary to consider that it differs for each spoke. The value 48 was obtained by computing the moment of inertia of a spoke perpendicular to OZ, multiplying by 20, deducting 10% for the part of spokes in counterbalance and boss, and then dividing the remainder by 2. This, of course, is only a reasonable approximation.

Problem 85. Compute the moment of inertia of a pair of drivers and their axle with respect to their axis of rotation. Use the data given above and assume the axle as cylindrical, the diameter being $9\frac{1}{2}''$ and the length $68''$. *Ans.* 2451.

Problem 86. Compute the moment of inertia of the pair of drivers and their axle, given in the preceding problem, with respect to an axis midway between the wheels and perpendicular to the axle. Consider the counterbalance of both wheels in such a position as to give a maximum moment of inertia and the distance between the centers of the wheels $60''$. *Ans.* 3445.

Problem 87. Find the moment of inertia of two cast-iron car wheels and their connecting steel axle with respect to (*a*) their axis of rotation, (*b*) an axis midway between the wheels and perpendicular to the axle. Consider the car wheels as composed of an outside tread, a circular web, and a hub; each part may be considered a hollow cylinder with the following dimensions: tread, outside radius 16″, inside radius 14″, width 5½″; web, outside radius 14″, inside radius 5½″, thickness 1.5″; hub, outside radius 5½″, inside radius 2½″, width 8″; axle (considered cylindrical), 5″ diameter and 7′ 3″ long. Distance between centers of wheels 60″. According to the assumption made above, the flange has been neglected, the web is considered a hollow disk, and the axle of uniform diameter throughout its length. The results will be approximately as follows: (*a*) 40, (*b*) 320.

Problem 88. The value 755 is the greatest value for the moment of inertia of a drive wheel with respect to a gravity axis in its plane. The least value will be with respect to an axis at right angles to this through the centers of gravity of the counterbalance and wheel. The student should compute this least moment of inertia.

Problem 89. In Problem 86 the drivers have been considered as having their cranks in the same plane. In practice they are 90° apart. Find the moment of inertia with respect to the axis stated when the wheels are so placed.

CHAPTER VIII

FLEXIBLE CORDS

66. Introduction. — A cord under tension due to any load may be considered as a rigid body. In the analysis of problems in which such cords are considered, the method of cutting or section may be used. Since the cord is flexible (requiring no force to bend it), it is easy to see that, no matter what forces are acting upon it, it must have at any point the direction of the resultant force at that point, and so must be under simple tension. If the cord is curved, as is the case where it is wrapped around a pulley, the resultant force is in the direction of the tangent.

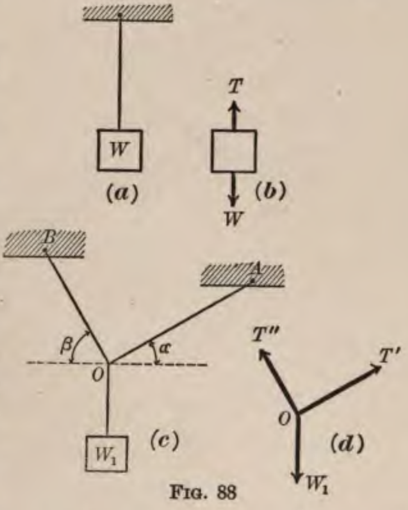


Fig. 88

Consider, as the simplest case, a weight W suspended by a cord, as shown in Fig. 88 (*a*). The forces acting on W are shown in (*b*) of the same figure. The cord has been considered cut and the force T, acting vertically upward, has been used to represent the tension. Summation of vertical forces $= 0$, gives $T = W$. When the weight W_1 is supported by two cords as in Fig. 88 (*c*), the cords *A* and *B* are under tension and may be cut. The system of forces acting on the point *O* is shown in (*d*), where T'' and T''' represent the tensions in the cords *A* and *B*, respectively. $\Sigma x = 0$ and $\Sigma y = 0$ give

$$T'' \cos \alpha = T''' \cos \beta$$

and
$$T'' \sin \alpha + T''' \sin \beta = W_1.$$

These two equations are sufficient to determine the unknown tensions T'' and T'''.

If two weights W_1 and W_2 are attached to the cord, as shown in the case of the cord *ABCD* (Fig. 89), each portion is under tension. Consider the cord cut at *A* and *D* and represent the tensions by T_1 and T_2 respectively. From $\Sigma x = 0$ and $\Sigma y = 0$ we have

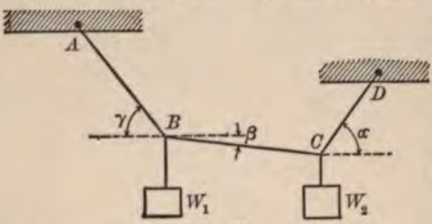


FIG. 89

$$T_1 \cos \gamma = T_2 \cos \alpha$$

and
$$T_1 \sin \gamma + T_2 \sin \alpha = W_1 + W_2.$$

A consideration of the forces acting at *B*, if we call the tension in the portion *BC*, T_3, gives, when the summation of the *x* and *y* forces are each put equal to zero,

$$T_1 \cos \gamma = T_3 \cos \beta$$

and
$$T_1 \sin \gamma - T_3 \sin \beta = W_1.$$

In a similar way, consider the forces acting on the point c, and we have

$$T_3 \cos\beta = T_2 \cos\alpha$$

and
$$T_3 \sin\beta + T_2 \sin\alpha = W_2.$$

Of the six equations given above only four are independent; consequently, of the six quantities T_1, T_2, T_3, α, β, and γ, two must be known in order to determine the other four.

In general, if there are n knots such as B and C of Fig. 89, with the weights W_1, W_2, W_3, W_4, etc., attached, it will be possible to get $n+2$ independent equations. These will be sufficient to determine the tension in each portion of the cord and its direction, provided the tension at A, say, and its direction are known. If the weights are close together, the curve takes more nearly the form of a smooth curve. Two special cases of this kind are discussed in this chapter in Art. 68 and Art. 69.

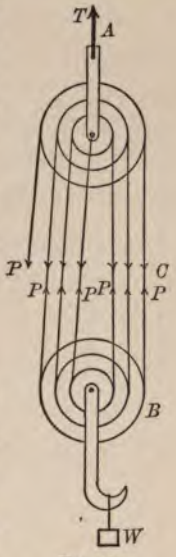


FIG. 90

67. Cords and Pulleys. — When a cord passes over a pulley, without friction, the tension is transmitted along its length undiminished. A weight W attached to a cord which passes vertically over a pulley is raised by a direct downward pull P on the other end of the rope. If there is no friction, P is equal to W. In the case of a system of pulleys, as shown in Fig. 90, the cord may be considered as under the same tension throughout and parallel to itself in passing from one sheave to the other. It is then possible to cut across the cords, just as was done in the case of the bridge truss, Problem 47, where the stress was along the member in each case. Cutting all the cords at C and considering all the forces acting on the sheave B, we get, calling the tension in the cord P,

$$6\,P = W,$$

or the tension in the cord is $W/6$. A consideration of the upper sheave gives $T = 7\,P = 7/6\,(W)$. The various cases of cords and pulleys that come up in engineering work may be taken up in a similar way, but in any case of cutting cords, it must be remembered that all cords attaching one part to another must be cut and the tension acting along the cords inserted before the principles of equilibrium can be applied. The consideration of the friction between cords and pulleys will be taken up in Chapter XIV.

68. Cord with Uniform Load Horizontally.—When a cord is suspended from two points A and B, Fig. 91 (a), and loaded with a uniform load horizontally in such a way that the points of attachment of the load to the cable are very close together, the cable takes the form of a

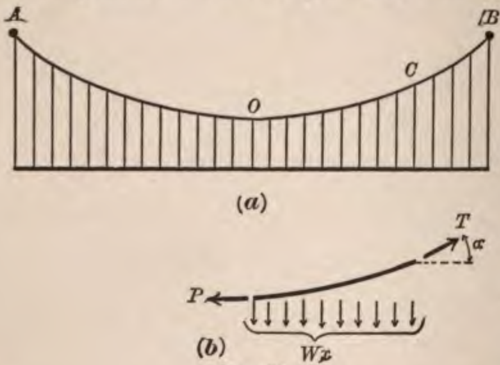


Fig. 91

continuous curve. The resultant tension in the cable in this case is in the direction of the cable at any point. Suppose the cable cut at the points O and C, where O is the middle point and C any point between O and B, and consider the forces acting on the cut portion. At C there is a tension T making an angle a with the horizontal, Fig. 91 (*b*). At O, the lowest point on the curve, the tension (P) is horizontal. The curve is supposed loaded with a uniform load of W pounds per linear foot, so that Wx represents the total horizontal loading. Writing down the equations $\Sigma x = 0$ and $\Sigma y = 0$, we obtain

$$P = T \cos a,$$

$$Wx = T \sin a,$$

and by division $\tan a = \frac{Wx}{P}$.

But $\tan a = \frac{dy}{dx}$, giving $\frac{dy}{dx} = \frac{Wx}{P}$.

Therefore, $y = \frac{Wx^2}{2P}$ or $x^2 = \frac{2P}{W}y$,

which is the equation of the curve taken by the cable under the assumed loading. This is a parabola. The deflection of the curve at any point can be found by putting in the value of x for that point and solving for y. If l equals length of span and d the maximum deflection at the center, then $d = \frac{Wl^2}{8P}$.

The length of the cable may be found by considering the formula

$$S_1 = \int_a^b ds,$$

where ds is measured along the curve. The length OB, or semi-length of the cable for a span l, is, since

$$ds^2 = dx^2 + dy^2,$$

$$OB = \int_a^b ds = \int_a^b \sqrt{dx^2 + dy^2} = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx.$$

From the equation of the curve $x^2 = \frac{2P}{W}y$, we have

$$\frac{dy}{dx} = \frac{Wx}{P}.$$

Therefore,

$$OB = \int_0^{\frac{l}{2}} \sqrt{1 + \frac{W^2x^2}{P^2}} \cdot dx = \frac{W}{P}\int_0^{\frac{l}{2}} \sqrt{\frac{P^2}{W^2} + x^2} \cdot dx$$

$$= \frac{W}{P}\left[\frac{x}{2}\sqrt{\frac{P^2}{W^2} + x^2} + \frac{P^2}{2W^2}\log_e\left(x + \sqrt{\frac{P^2}{W^2} + x^2}\right)\right]_0^{\frac{l}{2}}$$

$$= \frac{Wl}{4P}\sqrt{\frac{P^2}{W^2} + \frac{l^2}{4}} + \frac{P}{2W}\log_e\left[\left(\frac{l}{2} + \sqrt{\frac{P^2}{W^2} + \frac{l^2}{4}}\right) \div \frac{P}{W}\right].$$

Expanding the two terms of the above expression into infinite series and adding like terms, we may express the total length of the parabola in terms of l, W, and P:

$$\textbf{Total length} = l + \frac{W^2l^3}{24P^2} - \frac{W^4l^5}{768P^4} + \cdots$$

or in terms of l and d:

$$\textbf{Total length} = l + \frac{8d^2}{3l} - \frac{32d^4}{5l^3} + \cdots$$

In general, the convergence of either of the above series will be sufficiently rapid that only the first two terms need be used. In such cases they will be found convenient for computation.

The point of maximum tension in the cable is determined by considering the equation $P = T \cos \alpha$ or $T = \dfrac{P}{\cos \alpha}$. It is easy to see that T is greater, the smaller the value of $\cos \alpha$, that is, the larger the value of α, and this is greatest at the points of attachment A and B. This is the problem of the suspension bridge where the weight of the cable is neglected.

69. Equilibrium of a Flexible Cord Due to its Own Weight. — Where the loading along a flexible cord or cable is uniform, as in the case where the cable bends due to its own weight, the shape of the curve taken is no longer a parabola, as will be shown in what follows. Suppose a portion of the cable, OC, Fig. 92, with its load, cut free, and let the tensions in O and C (O is the middle point and C is any point between O and B) be P and T respectively. Then, $\Sigma x = 0$ and $\Sigma y = 0$ give $P = T \cos \alpha$ and $Ws = T \sin \alpha$, and from these two equations, by division, we have

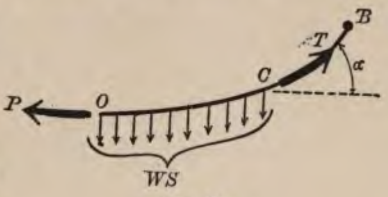


FIG. 92

$$\tan \alpha = \frac{Ws}{P} \text{ or } \frac{dy}{dx} = \frac{Ws}{P},$$

where s represents distance along the curve, and is related to x and y in such a way that $ds^2 = dx^2 + dy^2$. Eliminating dy between this and the previous equation, we obtain

$$dx = \frac{ds}{\sqrt{1 + \frac{W^2 s^2}{P^2}}} = \frac{P}{W} \cdot \frac{ds}{\sqrt{\frac{P^2}{W^2} + s^2}}.$$

Therefore,
$$x = \frac{P}{W}\log_e\left(s+\sqrt{\frac{P^2}{W^2}+s^2}\right)\Bigg]_0^s$$

$$= \frac{P}{W}\log_e\left[\frac{s+\sqrt{\frac{P^2}{W^2}+s^2}}{\frac{P}{W}}\right]. \qquad (a)$$

This gives a relation between x and s. A relation between y and s may be obtained by eliminating dx between the equations $\frac{dy}{dx}=\frac{Ws}{P}$ and $ds^2=dx^2+dy^2$. This gives

$$dy = \frac{s\,ds}{\sqrt{\frac{P^2}{W^2}+s^2}}.$$

Therefore,
$$y = \sqrt{\frac{P^2}{W^2}+s^2}\,\Bigg]_0^s = \sqrt{\frac{P^2}{W^2}+s^2}-\frac{P}{W}. \qquad (b)$$

Eliminating s between (a) and (b), we get the equation of the curve taken by the cable to be

$$y+\frac{P}{W}=\frac{P}{2W}\left(e^{\frac{Wx}{P}}+e^{-\frac{Wx}{P}}\right). \qquad (c)$$

This is the equation of the catenary with the origin at the vertex and the y-axis the axis of symmetry.

If the length of span is l, the maximum deflection of the cable at the center may be determined by substituting $x=\frac{l}{2}$ in (c). The semi-length of the cable may be found from (a) by substituting $x=\frac{l}{2}$ and solving for s. For this purpose (a) may be written in the exponential form;

remembering that $a^m = n$ may be written $m = \log_a n$, we then have

$$s = \frac{P}{2W}\left(e^{\frac{Wx}{P}} - e^{-\frac{Wx}{P}}\right). \qquad (d)$$

Since $P = T \cos \alpha$ or $T = \frac{P}{\cos \alpha}$, it is evident that the maximum tension occurs at the supports A and B.

70. Representations by Means of Hyperbolic Functions. — Equations (c) and (d) may be expressed in a simpler form by using the hyperbolic functions, remembering that the hyperbolic sine and cosine are expressed

$$\sinh \frac{Wx}{P} = \frac{e^{\frac{Wx}{P}} - e^{-\frac{Wx}{P}}}{2},$$

$$\cosh \frac{Wx}{P} = \frac{e^{\frac{Wx}{P}} + e^{-\frac{Wx}{P}}}{2}.$$

We have for equation (c)

$$y + \frac{P}{W} = \frac{P}{W} \cosh \frac{Wx}{P},$$

and for equation (d)
$$s = \frac{P}{W} \sinh \frac{Wx}{P}.$$

It is evident that for rapidity of computation of s and y, tables giving the values of the hyperbolic sine and cosine, for various values of $\frac{Wx}{P}$, would be convenient. For this reason a table is given in Appendix I, and the student is requested to use this table in solution of the problems.

Problem 90. A suspension bridge as shown in Fig. 93 has a span of 1200 ft. and the cable a maximum deflection at the center $d = 120$ ft. The weight of the floor is 2 tons per linear foot. Find the equation of the cable and the tension at O and at B. If the safe strength of cable is 75,000 lb. per square inch, find the area of wire section of cable necessary to support the floor.

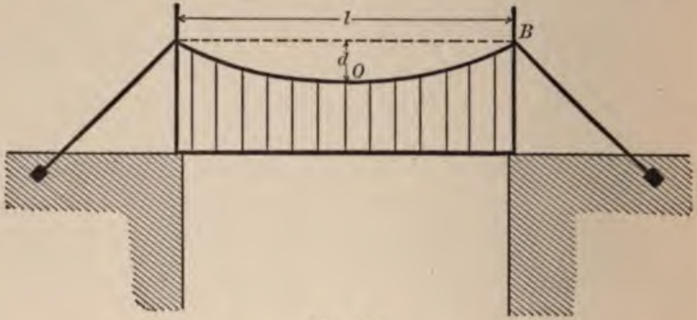


FIG. 93

Problem 91. Find the length of the cable in the preceding problem.

Problem 92. A flexible wire weighing $\frac{1}{2}$ lb. per foot is supported by two posts 200 ft. apart. The horizontal pull on the wire is 500 lb. Find the deflection at the center and the length of the wire.

Problem 93. What pull will be necessary in Problem 92 so that the greatest deflection will not be greater than 6 in.? What is the length of the wire for this case?

Problem 94. Find the tension in the wire of Problem 92 at the supports.

In practice use is often made of the fact that the exponential function may be expanded into an infinite series, so that

$$y + \frac{P}{W} = \frac{P}{W}\cosh\frac{Wx}{P}$$

may be written, remembering the meaning of $\cosh \frac{Wx}{P}$,

$$y + \frac{P}{W} = \frac{P}{2\,W}\left(2 + \frac{W^2x^2}{P^2} + \frac{W^4x^4}{3 \cdot 4\,P^4} + \cdots\right),$$

or
$$y = \frac{Wx^2}{2\,P} + \frac{W^3x^4}{2 \cdot 3 \cdot 4\,P^3} + \cdots.$$

In a similar way we may write

$$s = \frac{P}{2\,W}\left(\frac{2\,Wx}{P} + \frac{W^3x^3}{3\,P^3} + \frac{W^5x^5}{3 \cdot 4 \cdot 5\,P^5} + \cdots\right),$$

or
$$s = x + \frac{W^2x^3}{2 \cdot 3\,P^2} + \cdots$$

Since $\tan \alpha$ at the supports may be written $\tan \alpha = \sinh \frac{Wl}{2\,P}$, we may write it as the following series:

$$\tan \alpha = \frac{Wl}{2\,P} + \frac{W^3l^3}{48\,P^3} + \frac{W^5l^5}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 32\,P^5} + \cdots.$$

When the series are rapidly convergent, only the first terms need be used, so that

$$y = \frac{Wx^2}{2\,P} + \frac{W^3x^4}{2 \cdot 3 \cdot 4\,P^3},$$

$$s = x + \frac{W^2x^3}{2 \cdot 3\,P^2},$$

and
$$\tan \alpha = \frac{Wx}{P} + \frac{W^3x^3}{6\,P^3}.$$

When $y = d$, $x = \frac{l}{2}$, so that

$$d = \frac{Wl^2}{8P} + \frac{W^3l^4}{2 \cdot 3 \cdot 4 \cdot 16\,P^3},$$

or approximately $d = \frac{Wl^2}{8\,P}$, and so $P = \frac{Wl^2}{8\,d}$.

Also at the supports $\tan \alpha$ is approximately

$$\tan \alpha = \frac{Wl}{2P} = \frac{4d}{l}.$$

When $x = \frac{l}{2}$,

$$s = \frac{l}{2} + \frac{W^2 l^3}{48 P^2} = \frac{l}{2} + \frac{8 d^2}{6 l}.$$

The total length of the cable or wire may then be expressed as

$$\text{Total length} = l + \frac{8 d^2}{8 l}.$$

The student should make use of these formulæ in solving the preceding problems.

CHAPTER IX

MOTION IN A STRAIGHT LINE (RECTILINEAR MOTION)

71. Velocity. — *The velocity of a body is its rate of motion.* If the velocity is constant (uniform), it may be defined as the ratio of the distance passed over to the time spent in passing over that distance. If the velocity is variable, the velocity at any instant is the velocity that the body would have if at that instant the motion should become uniform. Speed is sometimes used instead of velocity, especially in speaking of the motion of machines or parts of machines. Speed, however, involves only the rate of motion without reference to the direction of motion, while velocity involves both rate of motion and the direction in which the motion takes place. Since constant velocity is the ratio of distance to time, it may be represented as

$$v = \frac{s}{t}.$$

The units for measuring velocity are those of distance and time, usually feet and seconds. Thus a body has a velocity of k feet per second or a train has a velocity of k miles per hour.

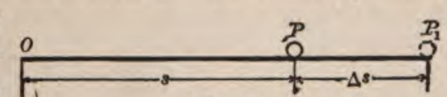


FIG. 94

A formula for expressing the relation between velocity, distance, and time for variable velocity may be derived by referring to Fig. 94. Suppose a body to have moved from O to P over the distance s with variable velocity. Let v be its velocity at P and t the time. In moving to another position P_1 distant Δs, the velocity changes by an amount Δv and the time by an amount Δt, so that at P_1

$$v + \Delta v = \frac{\Delta s}{\Delta t};$$

as Δv, Δs, and Δt approach zero as a limit,

$$v = \frac{ds}{dt};$$

that is, *the velocity is the first derivative of the distance with respect to time.*

72. Acceleration. — Acceleration may be defined as the *rate of change of velocity.* If the velocity changes by equal amounts in equal times, the acceleration is said to be *constant* or *uniform*, otherwise it is *variable.* Constant acceleration, then, is the ratio of the velocity to time; representing this acceleration by a_c, we have

$$a_c = \frac{v}{t}.$$

The units used are those of velocity and time, and since velocity is usually expressed in terms of feet and seconds or feet per second, acceleration is usually expressed in terms of feet per second per second. This is sometimes expressed as feet per square second or simply as feet per second, it being understood that the time must enter twice.

Since the acceleration equals the rate of change of velocity at any instant, we may write

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2},$$

where a represents the variable acceleration. Since

$$v = \frac{ds}{dt} \text{ and } a = \frac{dv}{dt},\ vdv = ads, \text{ by eliminating } dt.$$

73. Constant Acceleration. — When the acceleration is constant, we have the relation $dv = a_c dt$, a_c representing the constant value of a, and therefore

$$\int_{v_0}^{v} dv = a_c \int_0^t dt,$$

$$v = a_c t + v_0;$$

and since

$$v = \frac{ds}{dt},$$

$$\int_0^s ds = a_c \int_0^t t dt + v_0 \int_0^t dt,$$

or

$$s = \tfrac{1}{2} a_c t^2 + v_0 t.$$

In a similar way the relation

$$vdv = a_c ds$$

gives

$$\int_{v_0}^{v} vdv = a_c \int_0^s ds;$$

therefore

$$\frac{v^2}{2} - \frac{v_0^2}{2} = a_c s,$$

or

$$s = \frac{v^2 - v_0^2}{2a_c},$$

These equations of motion give the velocity in terms of time, the distance in terms of time, and the distance in terms of velocity.

74. Freely Falling Bodies. — Bodies falling toward the earth near its surface have a constant acceleration. It is usually represented by g and equals approximately 32.2 ft. per second squared. The value of g varies slightly with the height above the sea level and the latitude, but for the purposes of engineering it may usually be taken as 32.2. The equations of motion for such bodies are, then,

$$v = gt + v_0,$$
$$s = \tfrac{1}{2} gt^2 + v_0 t,$$
$$s = \frac{v^2 - v_0^2}{2g}.$$

If the body falls from rest, $v_0 = 0$, and the equations of motion become

$$v = gt,$$
$$s = \tfrac{1}{2} gt^2,$$
$$s = \frac{v^2}{2g}.$$

This latter is often written $v^2 = 2gh$, where $h = s$.

75. Body Projected vertically Upward. — When a body is projected vertically upward from the earth, the acceleration is constant and equals $-g$. If the velocity of projection is v_0, the equations of motion are

$$v = -gt + v_0 t,$$
$$s = -\tfrac{1}{2} gt^2 + v_0 t,$$
$$s = \frac{v^2 - v_0^2}{-2g}.$$

Problem 95. A body is projected vertically downward with an initial velocity of 30 ft. per second from a height of 100 ft. Find the time of descent and the velocity with which it strikes the ground.

Problem 96. A body falls from rest and reaches the ground in 6 sec. From what height does it fall, and with what velocity does it strike the ground?

Problem 97. A body is projected vertically upward and rises to the height of 200 ft. Find the velocity of projection v_0 and the time of ascent. Also find the time of descent and the velocity with which the body strikes the ground.

Problem 98. A stone is dropped into a well, and after 2 sec. the sound of the splash is heard. Find the distance to the surface of the water, the velocity of sound being 1127 ft. per second.

Problem 99. A man descending in an elevator whose velocity is 10 ft. per second drops a ball from a height above the elevator floor of 6 ft. How far will the elevator descend before the ball strikes the floor of the elevator?

Problem 100. In the preceding problem, suppose the elevator going up with the same velocity, find the distance the elevator goes before the ball strikes the floor of the elevator.

76. Newton's Laws of Motion. — Three fundamental laws may be laid down which embody all the principles in accordance with which motion takes place. These are the result of observation and experiment and are known as *Newton's Laws of Motion.*

First Law. Every body remains in a state of rest or of uniform motion in a straight line unless acted upon by some unbalanced force.

Second Law. When a body is acted upon by an unbalanced force, motion takes place along the line of action of the force, and the acceleration is proportional to the force applied.

Third Law. To every action of a force there is always an equal and opposite reaction.

The first law has already been made use of, and also the third law—see articles of Chapter II.

The second law states that in case the system of forces acting on the body is unbalanced, the motion is accelerated. Motion takes place in the direction of the resultant force with an acceleration proportional to the force. It also implies that each force of the system produces or tends to produce an acceleration in its own direction proportional to the force. That is to say, each force produces its own effect, regardless of the action of the other forces.

As a result of this latter fact, if a body is acted upon by a force P and the earth's attraction G, we have

$$P : G = a : g,$$

where G is the weight of the body, g the acceleration of gravity, and a the acceleration due to the force P. From this it follows that

$$\boldsymbol{P = \frac{G}{g} \cdot a = M \cdot a};$$

that is, the *accelerating* force equals the mass (see Art. 7) times the acceleration.

77. Motion on an Inclined Plane. — A body (see Fig. 95), of weight G, moves down an inclined plane, without friction under the action of a force $G \sin \theta$. The acceleration down the plane equals the accelerating force divided by the mass (see Art. 76) $= \dfrac{G \sin \theta}{\frac{G}{g}} = g \sin \theta$. The acceleration is constant.

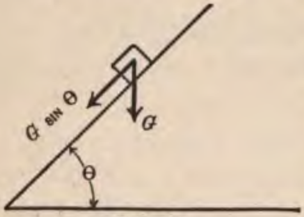


FIG. 95

The equations of motion for such a case, then, are (see Art. 73)

$$v = (g \sin \theta)\, t + v_0,$$

$$s = \frac{1}{2} g\, (\sin \theta)\, t^2 + v_0 t,$$

$$s = \frac{v^2 - v_0^2}{2\, g \sin \theta}.$$

If the body starts from rest down the plane, $v_0 = 0$. If it be projected up the plane with an initial velocity v_0, the acceleration equals $-g \sin \theta$.

Problem 101. A body is projected up an inclined plane which makes an angle of 60° with the horizontal with an initial velocity of 12 ft. per second. Neglecting friction of the plane, how far up the plane will the body go? Find the time of going up and of coming down.

Problem 102. A body is projected down the plane given in the preceding problem with a velocity of 20 ft. per second. How far will it go during the third second?

Problem 103. Suppose the body in the preceding problem meets a constant force of friction $F = 10$ lb. What will be the acceleration down the plane? How far will it go during the second second?

Problem 104. A boy who has coasted down hill on a sled has a velocity of 10 mi. per hour when he reaches the foot of the hill. He now goes on a horizontal, meeting a constant resistance of 25 lb. If the combined weight of the boy and sled is 75 lb., how far will he go before coming to rest?

Problem 105. Suppose that in the preceding problem the boy weighs 65 lb. and the sled 10 lb., and that the boy can exert a force of 20 lb. horizontally to keep him on the sled. Will the boy remain on the sled when the latter stops, or will he be thrown forward?

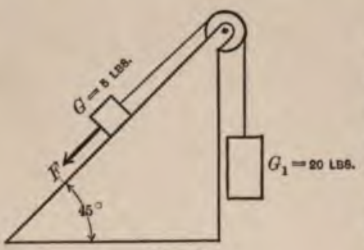


FIG. 96

Problem 106. A body whose weight $G = 5$ lb. is being drawn up an inclined plane as shown in Fig. 96 by the action of the weight $G = 20$ lb. Suppose the resistance offered by the plane $F = 10$ lb., and that G starts from rest. How far up the plane will G go in 6 sec.?

Problem 107. Two weights, $G_1 = 5$ lb. and $G_2 = 10$ lb., Fig. 97, attached to an inextensible cord which runs over a pulley, are acted upon by gravity; no friction; motion takes place. Find the tension in the cord, and the acceleration. Consider G_2 and G_1 separately with the forces acting upon them, and call the tension in the cord T. Then apply the principle "*accelerating force equals mass times acceleration.*"

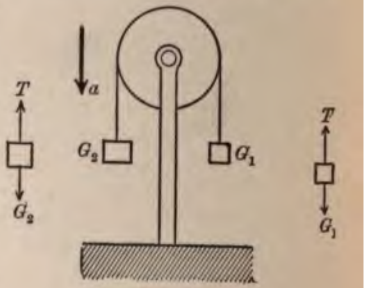


FIG. 97

Problem 108. An elevator, Fig. 98, whose weight is 2000 lb. is descending with a velocity, at one instant, of 2 ft. per second, and at the next second it has a velocity of 18.1 ft. per second. Find the tension T in the cable that supports the elevator.

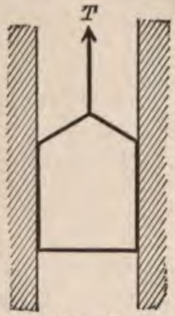


FIG. 98

Problem 109. Suppose the elevator in preceding problem going up with the same acceleration. Find the tension in the cable if the elevator starts from rest and attains its acceleration in 3 sec.

Problem 110. A man can just lift 200 lb. when standing on the ground. How much could he lift when in the moving elevator of the preceding problems, (*a*) when the elevator was ascending? (*b*) when descending?

Problem 111. Two weights, G and G', are connected by an inextensible flexible cord that passes over a frictionless pulley, as shown in Fig. 99. $G = 20$ lb., $G' = 100$ lb., and there is no friction on the plane. Find the tension in the cord and the acceleration of the two bodies.

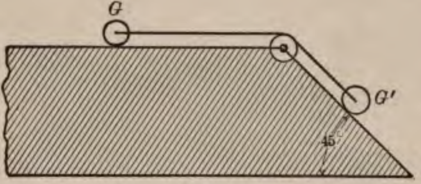


FIG. 99

Problem 112. A 30-ton car is moving with a velocity of 30 mi. per hour on a level track. The brakes refuse to work. How far will the car go after the power is turned off before coming to rest, if the friction is .01 of the weight of the car?

78. Variable Acceleration. — It has already been shown that $v = \frac{ds}{dt}$, $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$, and $vdv = ads$. These relations hold true no matter whether the acceleration is constant or variable. If the acceleration is *constant*, the equations of motion are those that have already been worked out (see Art. 73), and by simple substitution in these equations it is possible to find the velocity in terms of the time, the distance in terms of time, and the distance in terms of velocity.

If the acceleration is variable, it is necessary to work out the equations of motion for each case. This may be done, when it is known how a varies, by means of either of the equations,

$$a = \frac{d^2s}{dt^2},$$

or

$$vdv = ads.$$

The latter equation will usually give the beginner less difficulty.

79. Harmonic Motion. — Let it be supposed that a body is moved by an attractive force which varies as the distance. That is, the attractive force is proportional to the distance. Then the acceleration is also proportional to the distance.

Let the acceleration $= -ks.$

Then $$vdv = -ksds,$$

and $$\int_{v_0}^{v} vdv = -k\int_0^s sds;$$

therefore $$v^2 - v_0^2 = -ks^2,$$

where v_0 is the initial velocity when s equals zero and k is the factor of proportionality, determinable in any special case. This equation gives the relation between the velocity and distance. Since $v = \sqrt{v_0^2 - ks^2}$, it is evident that $v = 0$ when $\sqrt{k} \cdot s = v_0$. This means that the body comes to rest when s has reached a certain value, viz. $\frac{v_0}{\sqrt{k}}$. From the original assumption, $a = -ks$, it is seen that the acceleration is greatest when s is greatest, that is, when $s = \frac{v_0}{\sqrt{k}}$; and is least when s is least, that is, when $s = 0$.

To get the relation between distance and time, the equation $v = \sqrt{v_0^2 - ks^2}$ may be put in the form

$$\frac{ds}{\sqrt{v_0^2 - ks^2}} = dt,$$

from which, $$\frac{1}{\sqrt{k}} \sin^{-1}\frac{\sqrt{k} \cdot s}{v_0} = t,$$

or $$\frac{v_0}{\sqrt{k}} \sin \sqrt{k}t = s.$$

This relation between the distance and time shows that as t increases s changes in value from $\frac{v_0}{\sqrt{k}}$ to $\frac{-v_0}{\sqrt{k}}$, assuming all values between these limits, but never exceeding them, since $\sin \sqrt{k}t$ can never be greater than $+1$ or less than -1. The motion is, therefore, vibratory or periodic, and is known as *harmonic motion.* The complete period in this case is $\frac{2\pi}{\sqrt{k}}$.

The relation between velocity and time may be found for this case by differentiating the last equation with respect to time. Then,

$$v = v_0 \cos \sqrt{k}t.$$

This shows that v_0 is the greatest value of v.

This motion is usually illustrated by imagining a ball attached by means of two rubber bands or springs, since the force exerted by either of these is proportional to the elongation, to two pins, as shown in Fig. 100. Assuming that there is no friction and that the ball is displaced to a position B by stretching one of the rubber bands, when released it continues to move backward and forward with harmonic motion.

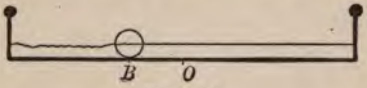


FIG. 100

Problem 113. Suppose the ball in Fig. 100, held by two helical springs, to have a weight of 10 lb. and that it is displaced 1 in. from O. The two springs are free from load when the body is at O. The springs are just alike, and each requires a force of 10 lb. to compress or elongate it 1 in. Find the time of vibration of the body and its velocity and position after $\frac{1}{2}$ sec. from the time when it is released. It has been found by experiment that the force necessary to compress or elongate a helical spring is proportional to the compression or elongation.

80. Motion with Repulsive Force Acting. — Suppose the force to be one of repulsion and to vary as the distance; then $a = ks$, and $vdv = ksds$, so that

$$v = \sqrt{v_0^2 + ks^2},$$

$$s = \frac{v_0}{2\sqrt{k}}\left(e^{\sqrt{k}\cdot t} - e^{-\sqrt{k}\cdot t}\right) = \frac{v_0}{\sqrt{k}} \sinh \sqrt{k}t\cdot$$

These equations show that as t increases s also increases and the body moves farther and farther away from the center of force. The motion is not oscillatory.

81. Motion where Resistance varies as Distance. — If a body whose weight is 644 lb. falls freely from rest through 60 ft. and strikes a resisting medium (a shaft where friction on the sides equals $2\ F = 10$ times the distance; see Fig. 101), since accelerating force equals mass times acceleration,

$$a = \frac{G - 2F}{M} = \frac{G - 10s}{\frac{G}{g}} = g - \frac{s}{2}.$$

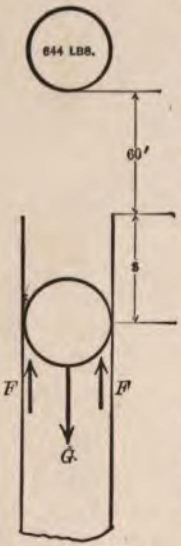


FIG. 101

It is required to find (a) the distance the body goes down the shaft before coming to rest; (b) the distance at which the velocity is a maximum; (c) the total time of fall; (d) the velocity at a distance of 10 ft. down the shaft. After striking the shaft the relation between velocity and distance is as follows:

$$\int_{v_0}^{v} v\,dv = \int_0^s \left(g - \frac{s}{2}\right) ds.$$

The remainder of the problem is left as an exercise for the student.

Problem 114. A ball whose weight is 32.2 lb. falls freely from rest through a distance of 10 ft. and strikes a 400-lb. spring, Fig. 102. Find the compression in the spring. It is to be understood that a 400-lb. spring is such a spring that 400 lb. resting upon it compresses it one inch, and 4800 lb. resting on it compresses it one foot, if such compression is possible. After the ball strikes the spring it is acted upon by the attraction of the earth and the resistance of the spring. The acceleration a is then $\dfrac{G - 4800\,s}{M}$, where s is measured in feet. The relation between velocity and distance is then obtained from the relation,

$$\int_{v_0}^{v} v\,dv = \int_0^s (g - 4800\,s)\,ds.$$

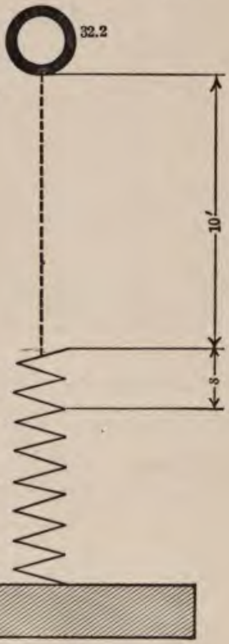


Fig. 102

Problem 115. A 20-ton freight car, Fig. 103, moving with a velocity of 4 mi. per hour strikes a bumping post. The 60,000-lb. spring of the draft rigging of the car is compressed. Find the compression s. Assume that the bumping post absorbs none of the shock.

Problem 116. Suppose the car in the preceding problem to be moving with a velocity of 4 mi. per hour, what should be the strength of the spring in the draft rigging so that the compression cannot exceed 2 in.?

Problem 117. After the spring in Problem 114 has been compressed so that the ball comes to rest, it begins to regain its original form. Find the time required to do this and the velocity with which the ball is thrown from the spring.

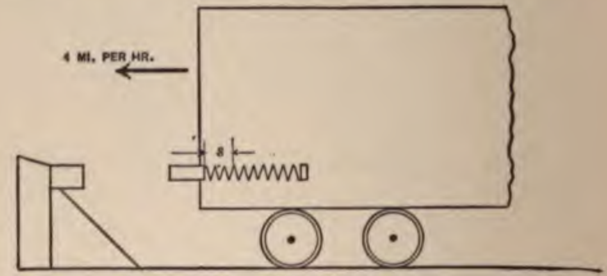


FIG. 103

82. Motion when Attractive Force varies inversely as Square Distance. — This is the case of motion, Fig. 104, when two bodies in space are considered, since in such cases the attractive force varies directly as the product of their masses and inversely as the square of the distance between them. The same attraction holds between two opposite poles of magnets or between two bodies charged oppositely with electricity.

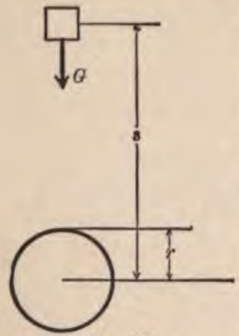


FIG. 104

Suppose the acceleration $= \frac{-k}{s^2}$ and that the velocity is zero.

Then, $\int_v^0 v dv = -\int_s^{s_0} \frac{k}{s^2} ds$,

so that
$$v = \sqrt{\frac{2k}{s_0}} \frac{\sqrt{s_0 s - s^2}}{s},$$

and
$$t = \sqrt{\frac{s_0}{2k}} \left[\sqrt{s_0 s - s^2} - \frac{s_0}{2} \text{vers}^{-1} \frac{2s}{s_0} + \frac{\pi s_0}{2} \right].$$

The time required to reach the center of attraction O from the position of rest is obtained by putting $s = 0$. This gives $t = \frac{\pi}{\sqrt{k}} \left(\frac{s_0}{2} \right)^{\frac{3}{2}}$.

It is seen that when $s = 0$ the velocity is infinite, and therefore the body approaches the center of attraction with increasing velocity and passes through the center, to be retarded on the other side until it reaches a distance $- s_0$. The motion will be oscillatory.

If one of the bodies is the earth, of radius r, and the other is a body of weight G falling toward it, the equations just derived hold true. In this case it is possible to determine k. The attraction on the body at the surface of the earth is G, and at a distance s is F, so that $F = G\left(\frac{r^2}{s^2} \right)$. The acceleration is therefore $\frac{-F}{M} = -g\left(\frac{r^2}{s^2} \right)$. This gives k, then, equal to $r^2 g$.

Substituting these values in the above equation, we find

$$v = \sqrt{\frac{2gr^2}{s_0}} \frac{\sqrt{s_0 s - s^2}}{s}.$$

When $s = r$ at the earth's surface,

$$v = \frac{\sqrt{2gr}}{s_0} \sqrt{(s_0 - r)} = \sqrt{2gr} \sqrt{\left(1 - \frac{r}{s_0}\right)}$$

If $$s_0 = \infty,\ v = \sqrt{2\,gr}.$$

But this is a value of v that cannot be obtained, since s_0 cannot be infinite. So that the velocity is always less than $\sqrt{2\,gr}$. It is interesting to notice here that if a body were projected from the earth with a velocity greater than $\sqrt{2\,gr}$, it would never return, provided there were no atmospheric resistance. Substituting $g = 32.2$ and $r = 3963$ mi.,

$$v = 6.7 \text{ mi. per second.}$$

This is the greatest velocity that a body could possibly acquire in falling to the earth, and a body projected upward with a greater velocity would never return (neglecting resistance).

If the body falls to the earth from a height h, the velocity acquired may be obtained from the foregoing by putting $s = r$ and $s_0 = h + r$; then

$$v = \sqrt{\frac{2\,grh}{r + h}}.$$

If h is small compared to r, this may be written, without serious error,

$$v = \sqrt{2\,gh},$$

which is the formula derived for a freely falling body in Art. 74.

83. Motion of a Body through the Atmosphere. —When a body such as a raindrop moves through the air, the resistance varies approximately as the square of the velocity. Suppose a body of weight G projected vertically upward in such a medium and let the resistance be $R = Ma = -G - kv^2$,

$$a = -g\left(1 - \frac{k}{G}v^2\right).$$

And the relation between velocity and distance is expressed by the equation

$$vdv = -g\left(1+\frac{k}{G}v^2\right)ds,$$

or

$$\int_{v_0}^{v}\frac{vdv}{1+\frac{k}{G}v^2} = -g\int_0^s ds\,;$$

$$\frac{G}{2k}\log\left[\frac{1+\frac{k}{G}v^2}{1+\frac{k}{G}v_0^2}\right] = -gs.$$

This gives the relation between velocity and distance. It is left as a problem for the student to determine the relation between distance and time for this case. Find the greatest height to which a body will rise and the velocity with which it strikes the ground upon returning. Compare this velocity with the velocity of projection.

84. Relative Velocity. — When we speak of the velocity of a body, it is understood that we mean the velocity of the body relative to the earth, more particularly the point on the earth from which the motion is observed. Since the earth is in motion, it is evident that velocity as generally spoken of is not absolute velocity, and since there is nothing in the universe that is at rest, all velocities must be relative. In everyday life, however, we think of velocities referred to any point on the surface of the earth as being absolute.

A person walking on the deck of a boat, for example, has a velocity relative to the earth and a velocity relative to the boat; the former is usually spoken of as the absolute velocity, and the latter the relative velocity. Or, suppose the case of a man standing on the deck of a boat moving south with a velocity v while the wind blows from the east with a velocity v_1. It is required to find the velocity of the wind with respect to the man, or, in other words, the apparent direction and velocity of the wind as observed by the man. Referring to Fig. 105, we represent the velocity (see Art. 85) of the boat with respect to the earth by v, and the velocity of the wind with respect to the earth by v_1, then V represents the velocity of the wind with respect to the man; that is, the wind appears to the man to be coming from the southeast. The velocity V was obtained by reversing the arrow representing the velocity of the boat and finding the resultant of this reversed velocity and the velocity v_1 of the wind.

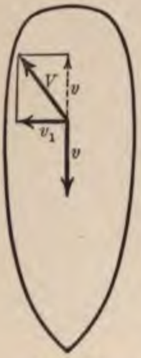


Fig. 105

If v_1 be considered as the velocity with respect to the earth of a man walking across the deck of a steamer moving with a velocity v, then V represents the velocity of the man with respect to the boat.

Problem 118. An ice boat is moving due north at a speed of 60 mi. per hour, the wind blows from the southwest with a velocity of 20 mi. per hour. What is the apparent direction and velocity of the wind as observed by a man on the boat?

Problem 119. A man walks in the rain with a velocity of 4 mi. per hour. The rain drops have a velocity of 20 ft. per second in a direction making 60° with the horizontal. How much must the man incline his umbrella from the vertical in order to keep off the rain: (*a*) when going against the rain, (*b*) when going away from the rain? If he doubles his speed, what change is necessary in the inclination of his umbrella in (*a*) and (*b*)?

Problem 120. The light from a star enters a telescope inclined at an angle of 45° with the surface of the earth. The velocity of light is 186,000 mi. per second and the earth (radius 4000 mi.) makes one revolution in 24 hr. What is the actual direction of the star with respect to the earth? This displacement of light due to the velocity of the earth and the velocity of light is known as *aberration of light.*

Problem 121. A man attempts to swim across a river, 1 mi. wide, which is flowing at the rate of 4 mi. per hour. If he can swim at the rate of 3 mi. per hour, what direction must he take in swimming in order to reach a point directly across on the opposite shore?

Problem 122. A train is moving with a speed of 60 mi. per hour, another train on a parallel track is going in the opposite direction with a speed of 40 mi. per hour. What is the velocity of the second train as observed by a passenger on the first?

Problem 123. A man in an automobile going at a speed of 40 mi. per hour is struck by a stone thrown by a boy. The stone has a velocity of 30 ft. per second and moves in a direction perpendicular to the direction of motion of the automobile. With what velocity does the stone strike the man?

Problem 124. A locomotive is moving with a velocity of 40 mi. per hour. Its drive wheels are 80 in. in diameter. What is the tangential velocity of the upper point of the wheels with respect to the frame of the locomotive? What is the tangential velocity of the lowest point?

CHAPTER X

CURVILINEAR MOTION

85. Representation of Velocity and Acceleration. — It has been shown (Art. 71) that velocity is measured in terms of feet per second, miles per hour, or in general in terms of the units of distance and units of time. The velocity is, moreover, in a given direction and may accordingly be represented by an arrow just as forces may be so represented. It follows then that *velocity* arrows may be added algebraically if parallel and if such addition is not inconsistent with the problem. They may be resolved into components or combined to form resultants (see Art. 11 and Art. 12). In case the body moves in a curve it is often desirable, instead of dealing with the resultant velocity along the tangent, to deal with the components of that velocity along the two coördinate axes. Thus if v is the velocity along the tangent (Fig. 106), $\boldsymbol{v_x = v \cos \alpha}$ and $\boldsymbol{v_y = v \sin \alpha}$ are

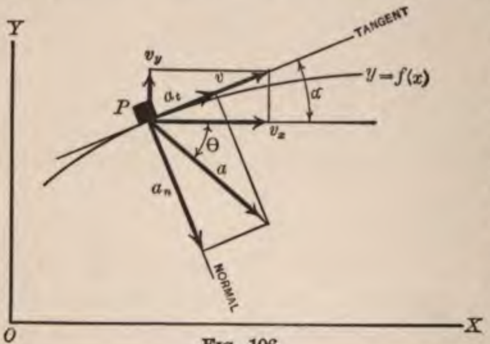


FIG. 106

the component velocities along the axes x and y respectively. In a similar way if we know the velocity of a body along the x-axis, v_x, and the velocity along the y-axis, v_y, we find the resultant velocity to be $v=\sqrt{v_x^2+v_y^2}$ and its direction with the x-axis such that $\tan\alpha=\frac{v_y}{v_x}$.

Accelerations have been seen to be measured in terms of units of distance and units of time, in particular in terms of feet per (second)^2 (see Art. 72). An acceleration may be represented by an arrow, the length of the arrow representing the number of feet per (second)^2 and the direction of the arrow giving the direction of the acceleration. Since arrows represent accelerations, the acceleration arrows may be treated just as velocity arrows. That is, they may be added algebraically if parallel, or added and subtracted geometrically if intersecting. Or we may say that the *parallelogram law* holds for accelerations. Referring to Fig. 106, it is seen that the resultant acceleration, a, of the body moving in the curve $y=f(x)$ is directed toward the concave side of the curve in the direction of the resultant force. This is evident from Newton's Law, which states that *the acceleration is proportional to the resultant force and in the same direction.* Let a be the resultant acceleration, then the accelerations along the two axes are $\boldsymbol{a_x=a\cos\theta}$ and $\boldsymbol{a_y=a\sin\theta}$, respectively. In a similar way if we know the accelerations along the two axes, a_x and a_y, the resultant acceleration $a=\sqrt{a_x^2+a_y^2}$, and its direction is given by the equation

$$\tan\theta=\frac{a_y}{a_x}.$$

86. Tangential and Normal Accelerations.—Suppose a body (Fig. 106) moves in any curve $y=f(x)$ and that at a certain point P it has a resultant velocity v and a resultant acceleration a, v acts along the tangent, which at this point makes an angle α with the x-axis and a acts along the line of action of the resultant force on the concave side of the curve. It is seen from the figure that $v_x = v\cos\alpha$, $v_y = v\sin\alpha$, $a_x = a\cos\theta$, $a_y = a\sin\theta$, so that $v=\sqrt{v_x^2+v_y^2}$ and $a=\sqrt{a_x^2+a_y^2}$.

It is usually convenient in curvilinear motion to consider the acceleration along the tangent and normal; that is, a_t and a_n. Since the tangent and normal are at right angles, it is evident that $a=\sqrt{a_t^2+a_n^2}$. When it is remembered that v acts along the tangent, it is evident that the tangential acceleration $a_t=\frac{dv}{dt}$, or,

$$a_t=\frac{d}{dt}\sqrt{v_x^2+v_y^2}=\frac{d}{dt}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$$

$$=\frac{1}{\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}}\left(\frac{d^2x}{dt^2}\frac{dx}{dt}+\frac{d^2y}{dt^2}\frac{dy}{dt}\right),$$

$$=\frac{1}{v}\left(a_xv_x+a_yv_y\right)=a_x\cos\alpha+a_y\sin\alpha,$$

since $$\frac{v_x}{v}=\cos\alpha \text{ and } \frac{v_y}{v}=\sin\alpha.$$

It now remains to find the normal acceleration. It has been shown that $a=\sqrt{a_n^2+a_t^2}$; therefore, $a_n=\sqrt{a^2-a_t^2}$. Substituting the value of $a^2=(a_x^2+a_y^2)$, and the value of a_t just found, we have as the value of the normal acceleration,

$$a_n = a_y \cos \alpha - a_x \sin \alpha.$$

The *normal force* and *tangential force* may now be found by multiplying by the mass M, giving,

$$\textit{Normal force} = Ma_y \cos \alpha - Ma_x \sin \alpha;$$

$$\textit{Tangential force} = Ma_x \cos \alpha + Ma_y \sin \alpha.$$

It is usually, however, more convenient to have the normal and tangential forces expressed in terms of the velocity. Since $a_t = \frac{dv}{dt}$, the tangential force $= M\frac{dv}{dt}$. To express the normal acceleration in terms of velocity it is necessary to write,

$$a_n = \left(\frac{d^2y}{dt^2}\frac{dx}{dt} - \frac{d^2x}{dt^2}\frac{dy}{dt}\right)\frac{1}{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}$$

$$= \frac{\frac{d^2y}{dt^2}\frac{dx}{dt} - \frac{d^2x}{dt^2}\frac{dy}{dt}}{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]^{\frac{3}{2}}}\left(\frac{ds}{dt}\right)^2$$

$$= \frac{1}{\rho}\left(\frac{ds}{dt}\right)^2 = \frac{v^2}{\rho}. \qquad \text{(See note.)}$$

NOTE. Since y is a function of x and both x and y are functions of t, we may write,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{\frac{d^2y}{dt^2}\frac{dx}{dt} - \frac{d^2x}{dt^2}\frac{dy}{dt}}{\left(\frac{dx}{dt}\right)^3}$$

The expression for the radius of curvature of a curve whose equation is $y = f(x)$ is

$$\rho = \frac{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dt^2}\frac{dx}{dt} - \frac{d^2x}{dt^2}\frac{dy}{dt}}.$$

The normal force and tangential force may now be written,

$$\textbf{Normal force} = \frac{Mv^2}{\rho};$$

$$\textbf{Tangential force} = M\frac{dv}{dt}.$$

For all curves except the circle ρ the radius of curvature varies from point to point. In the circle, however, it is the radius, and is therefore constant. In this case the normal force is usually called the centripetal force.

87. Uniform Motion in a Circle. — A body moving with constant velocity in the circumference of a circle is acted upon by one force, the normal or centripetal force, and this equals $\frac{Mv^2}{r}$. That this is true is evident when it is remembered that the tangential velocity is constant, thus making the tangential acceleration zero. An illustration of uniform motion in a circle is seen in the case of the simple governor shown in Fig. 107. When the velocity is constant, then α, h, and r are constant. Let T be the tension in the rod supporting the ball, then, since there is no vertical motion $\Sigma y = 0$, so that $T \cos \alpha = G$. Considering the normal force, we have $T \sin \alpha = \frac{Mv^2}{r}$, so that $\tan \alpha = \frac{v^2}{gr}$. From these equations T may be found for any values of α and r.

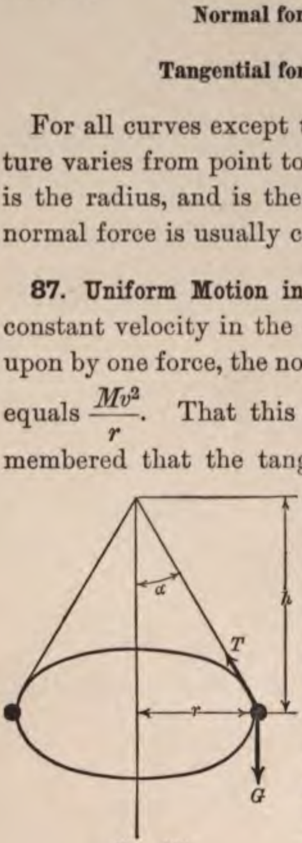


FIG. 107

Problem 125. The weighted governor shown in Fig. 108 is rotated at such a speed that $\alpha = 30°$. Find the forces acting on the longer rods and the stress in the shorter rods. The connections are all pin connections.

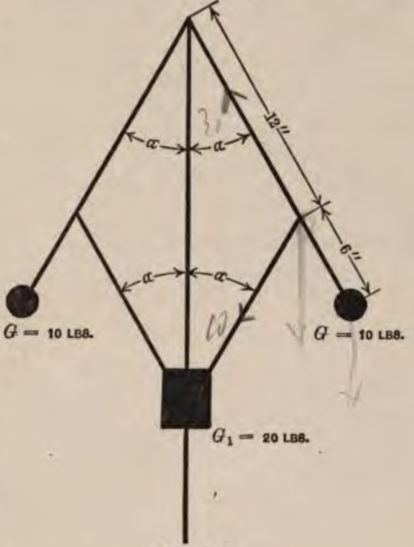


FIG. 108

Problem 126. A type of swing is shown in Fig. 109. A revolving central post supported by wires A and B carries six cars G, each suspended from cross arms D by means of cables 50 ft. long. When the swing is at rest, the cars hang vertically and $\alpha = 0$; as the speed of rotation increases, α becomes larger. Suppose the car and its load of four passengers to weigh 1000 lb., and the speed to be such that $\alpha = 30°$, find the tension in the cables supporting the cars. Assume that a single car is carried by one cable.

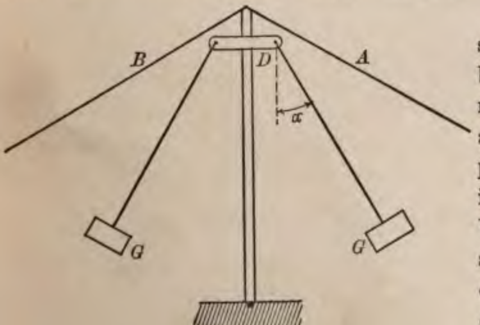


FIG. 109

Problem 127. The same principle that has been seen to hold for motion in a circle enables us to solve the problem that comes up in railroad work. When a train goes around a curve, it is desirable to have the outer rail raised sufficiently so that the wheel pressure will be normal to the rails. It is really the same problem as Problem 126, where the sustaining cable is replaced by a track (see Fig. 110). Let r be the radius of curvature, v the velocity of the car, of weight G. Show that the superelevation of the outer rail is given by $\tan\alpha = \frac{v^2}{gr}$, and so

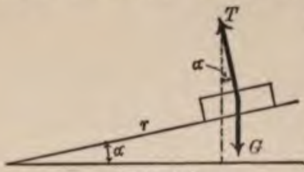


FIG. 110

$$h = \frac{dv^2}{gr},$$

where d is the distance between the rails in feet, v the velocity in ft. per second, g is 32.2, r is the radius of curvature in feet, and h is the superelevation of the outer rail in feet. This height may be expressed, approximately, as follows:

$$h = \frac{v_1^2}{3r},$$

where h and r are in feet and v_1 is the velocity in miles per hour. Here d has been taken as 4.9 ft. Using this latter formula, the following table for the superelevation of the outer rail has been constructed:

ELEVATION (h) IN FEET FOR GIVEN RADIUS IN FEET

VELOCITY MILES PER HOUR	5730	2865	1910	1632	1146	955
20	.02	.05	.07	.09	.12	.14
30	.05	.10	.16	.21	.26	.31
40	.09	.19	.28	.37	.46	.56
50	.15	.29	.44	.58	.73	.87
60	.21	.42	.63	.84	1.04	1.25

88. Simple Circular Pendulum. — The simple circular pendulum consists of a weight G suspended by a string without weight, of length l, in such a way that it is free to move in a circle in a vertical plane due to the action of gravity (see Fig. 111). Let B be such a position of the pendulum that its height above the horizontal is h, and c any other position designated by the coördinates x and y. Let the weight be G and the tension in the string T. These are the only forces acting on the body C. The only forces that can produce motion in the circle are those that are tangent to the circle OB. The force T is normal to the circle and so has no tangential component. The force G has a tangential component $-G \sin \alpha$. The equation of motion is, therefore,

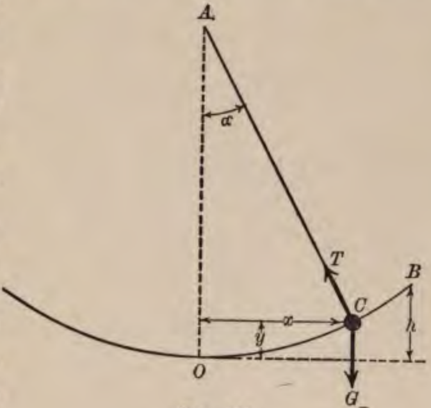


FIG. 111

$$vdv = ads = -g \sin \alpha ds,$$

where $\alpha = \frac{\text{arc}\,(oc)}{l} = \frac{s}{l}$, where s denotes distance along the curve.

This equation, as it stands, leads to a complicated relation between distance and time. If, however, the angle α is sufficiently small, so that we may replace $\sin \alpha$ by α, we may write

$$\int_0^v vdv = \frac{-g}{l}\int_{s_h}^{s} sds.$$

Integrating with respect to s, we have

$$v^2 = \frac{g}{l}s_h^2 - \frac{g}{l}s^2,$$

where $v=0$, $s=s_h$, where s_h is length of curve OB. Solving for the velocity, we have

$$v=\sqrt{\frac{g}{l}(s_h^2-s^2)}$$

or
$$\frac{ds}{dt}=\sqrt{\frac{g}{l}(s_h^2-s^2)},$$

which may be put in the form

$$t=\int\frac{ds}{\sqrt{\frac{g}{l}(s_h^2-s^2)}}=\sqrt{\frac{l}{g}}\cos^{-1}\left(\frac{s}{s_h}\right)+c,$$

where c is a constant of integration.

This equation may be written,

$$s=s_h\cos\left[\sqrt{\frac{g}{l}}(t-c)\right];$$

it represents the relation between distance and time.

When $s=s_h$, $t=t_h$, so that

$$\left[\sqrt{\frac{g}{l}}(t_h-c)\right]=0;$$

therefore, $c=t_h$.

It is evident that s is a periodic function of the time, and that it repeats itself at intervals of time t, such that

$$t=2\pi\sqrt{\frac{l}{g}}.$$

This value t_h represents the time taken by the body from leaving the position B until its return. One half of this value

$$t_h=\pi\sqrt{\frac{l}{g}}$$

is designated as the *period of vibration.*

In general, when the angle is not small, the equation $vdv = -g \sin \alpha ds$ becomes, since $ds \sin \alpha = dy$,

$$\int_0^v vdv = -g \int_h^y dy.$$

Integrating, we have $v^2 = -2gy \Big]_h^y = 2g(h-y)$.

It will be seen that this result is the same as if the body had fallen freely through a height $h-y$ (see Art. 74). The value for v is evidently true whatever be the vertical curve in which the body moves, providing the only forces acting on the body are the force of gravity and another force normal to the curve. The foregoing fact leads to the statement, *in descending along any curve without friction, from a height h to any other height y a body will have the same velocity as if it fell freely through the height $h-y$.* This fact is often made use of in mechanical problems.

The summation of forces normal to the path gives

$$T - G \cos \alpha = \frac{Mv^2}{l},$$

and, therefore,

$$T = G \cos \alpha + \frac{Mv^2}{l},$$

which gives a value for the tension in the cord. Since v is greatest when $\alpha = 0$, it is seen that the greatest tension in the cord occurs when the pendulum is vertical.

If now we make use of the fact that the body moves in a circle whose equation is $x^2 + y^2 - 2ry = 0$, and remember that $ds^2 = dx^2 + dy^2$ for any curve, we may write

$$v^2 = \left(\frac{ds}{dt}\right)^2,$$

and, therefore, $\left(\frac{ds}{dt}\right)^2=\frac{dx^2+dy^2}{dt^2}=2\,g(h-y)$,

or
$$dt=\frac{\sqrt{dx^2+dy^2}}{\sqrt{2\,g\,(h-y)}}.$$

But from the equation of the curve
$$dx=\frac{(y-r)dy}{\sqrt{2\,ry-y^2}},$$

so that
$$dx^2=\frac{(y-r)^2dy^2}{(2\,ry-y^2)},$$

and
$$dt=\frac{-\,rdy}{\sqrt{2\,ry-y^2}\cdot\sqrt{2\,g\,(h-y)}}.$$

The integral of the expression on the right-hand side of the equation is not expressible in terms of ordinary algebraic or trigonometric functions, but must be expressed in terms of the *elliptic functions*. The student may not be familiar with such functions, so that we shall express it approximately by means of an infinite series. This series will be sufficiently rapidly convergent if the radius of the circle is large and the distance OB is small. Using the minus sign in the numerator, since t is a decreasing function of s, we may write

$$t=\sqrt{\frac{r}{g}}\int_0^h\frac{dy}{\sqrt{hy-y^2}}\left(1-\frac{y}{2\,r}\right)^{-\frac{1}{2}}$$

$$=\sqrt{\frac{r}{g}}\int_0^h\left[1+\frac{1}{2}\left(\frac{y}{2\,r}\right)+\frac{1}{2}\cdot\frac{3}{4}\left(\frac{y}{2\,r}\right)^2+\cdots\right]\frac{dy}{\sqrt{hy-y^2}}$$

$$=\pi\sqrt{\frac{r}{g}}\left[1+(\tfrac{1}{2})^2\frac{h}{2\,r}+\left(\frac{1\cdot3}{2\cdot4}\right)^2\left(\frac{h}{2\,r}\right)^2+\left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^2\left(\frac{h}{2\,r}\right)^3+\text{etc.}\right].$$

For very small values of h we may neglect the terms containing h. The result then becomes

$$t=\pi\sqrt{\frac{l}{g}},$$

since $r=l$, and this is the same result that was obtained before.

This means that for small values of α, not greater than 4°, the time of vibration of a simple circular pendulum is a *constant;* that is, *the oscillations* are *isochronal.*

It is seen that the time of vibration of a simple pendulum varies as the square root of its length, for any locality on the earth. In order to get a pendulum that will beat seconds it is necessary to place $t=1$. Knowing the value of g for the locality, the proper length may be determined. If we measure the length of a pendulum and its period we may calculate the value for g for any locality. This is the easiest and most accurate way of determining g.

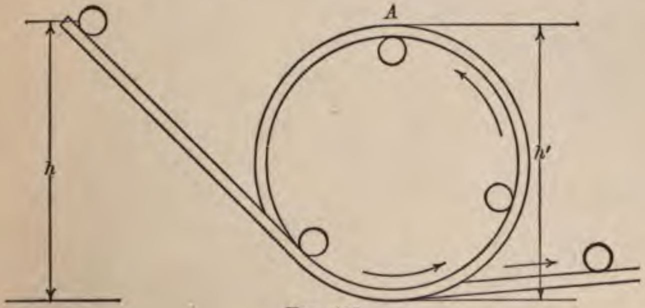


FIG. 112

Problem 128. The centrifugal railway (Fig. 112), or "loop the loop," is a common example of a simple circular pendulum, where the effect of the string is replaced by a track. If we neglect friction, the only forces acting on the car are the force of gravity and the normal pressure of the track. Suppose the car starts from rest at a height, h. What must be the relation between h and h', so that the car will pass the point A without leaving the track.

Hint. The velocity at the lowest point, $v^2 = 2\,gh$, is the same as the velocity with which the car comes down. The centrifugal force must be great enough at A to overcome G, the weight of the car $(h = \frac{5}{4}h')$.

Problem 129. In the simple pendulum find the value of y in terms of h for which the tension in the string is the same as when the pendulum hangs at rest.

Problem 130. A pendulum vibrates seconds at a certain place and at another place it makes 60 more vibrations in 12 hours. Compare the values of g for the two places.

89. Cycloidal Pendulum. — It has been found that a pendulum may be obtained whose period of vibration is constant by allowing the string to wrap itself around a cycloid as shown in Fig. 113. The pendulum hangs from the point A. AB and AC are cycloidal guides around

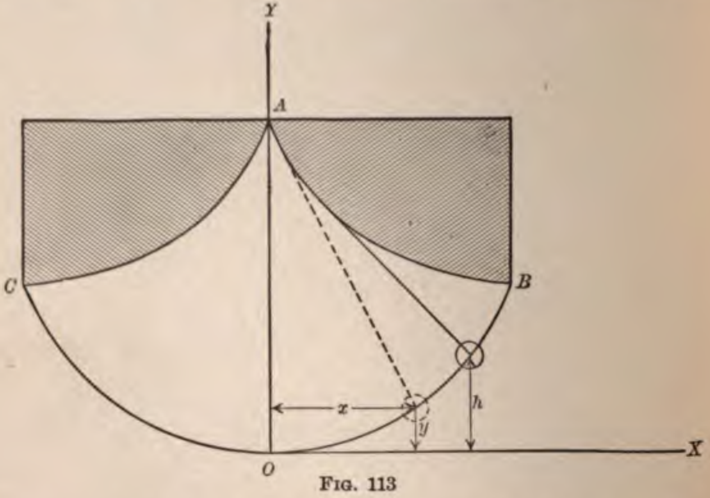


Fig. 113

which the string wraps as the pendulum swings. This causes the length of the pendulum to continually change and the pendulum "bob" to move in another cycloidal curve *COB*. The equation of this curve referred to the axes x and y is

$$x = \frac{l}{4}\operatorname{vers}^{-1}\left(\frac{4y}{l}\right) + \sqrt{\frac{ly}{2} - y^2}.$$

In Art. 88 it was seen that $v^2 = 2g(h-y)$ represented the velocity of a body moving in a vertical curve when only the force of gravity and a force normal to the path of the curve acted. We may make use of the equation in this case, since the same conditions exist. We may write

$$dt = \frac{\sqrt{dx^2 + dy^2}}{\sqrt{2g(h-y)}}, \text{ since } v = \frac{ds}{dt}.$$

From the equation of the curve, we find

$$dx = \frac{\left(\frac{l}{2} - y\right)dy}{\sqrt{\frac{ly}{2} - y^2}},$$

so that

$$\int_0^t dt = \sqrt{\frac{l}{4g}}\int_0^h \left(\frac{-dy}{\sqrt{hy - y^2}}\right)$$

taking the negative sign, since t is a decreasing function of s.

Therefore $$t = \sqrt{\frac{l}{4g}}\left[\operatorname{vers}^{-1}\frac{2y}{h}\right]_0^h = \pi\sqrt{\frac{l}{4g}}.$$

The whole time of vibration is twice this value, so that the time of vibration

$$t = \pi\sqrt{\frac{l}{g}}.$$

This expression is independent of h, so that all vibrations are made in the same time. The motion is therefore isochronal.

Problem 131. A body of mass M slides from rest down a cycloid from the position B (Fig. 113) without friction. What is its velocity when $y = \frac{h}{2}$? Show that this is its maximum velocity.

Problem 132. Find the position of a cycloidal pendulum where the tension in the string is greatest. What is the velocity of the bob at the point O (Fig. 113)?

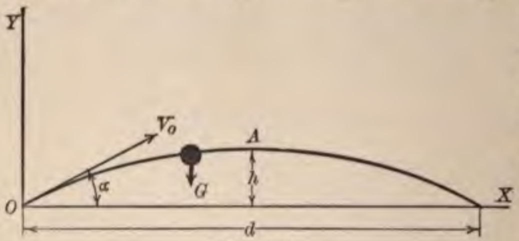


FIG. 114

90. Motion of Projectile in Vacuo. — A method, slightly different from the preceding, of dealing with a problem of curvilinear motion, is illustrated in the present article. It is desired to find the path taken by a body projected with a velocity v_0 at an angle of elevation α, when the resistance of the air is neglected (see Fig. 114). In this case, since there is no horizontal force acting on the body $a_x = 0$, so that,

$$\frac{d^2x}{dt^2} = 0$$

and

$$\frac{dx}{dt} = \text{constant} = v_0 \cos \alpha,$$

therefore, $x = v_0 \cos \alpha \, (t)$.

In a similar way we know that the vertical acceleration $a_y = -g$, since the only force acting is G.

Then, $$\frac{d^2y}{dt^2} = -g$$

and $$\frac{dy}{dt} = -gt + \text{constant.}$$

This equation may be rewritten

$$v_y = -gt + \text{constant.}$$

To determine this constant of integration, we put $t = 0$,

and $$v_y = v_{0y} = v_0 \sin \alpha;$$

therefore $$\frac{dy}{dt} = -gt + v_0 \sin \alpha$$

and $$y = -\tfrac{1}{2}gt^2 + v_0 \sin \alpha\,(t).$$

Eliminating t between the equations in x and y, we get

$$y = x \tan \alpha - \frac{gx^2}{2\,v_0^2 \cos^2 \alpha}$$

as the equation of the path of the projectile. This is evidently a parabola, with its axis vertical and its vertex at A.

Range. To find the range or horizontal distance d we put $y = 0$; then $x = 0$ and $x = d$,

so that $$d = \frac{v_0^2 \sin 2\,\alpha}{g}.$$

From this it is clear that the greatest range is given when $\alpha = 45°$, since then $d = \frac{v_0^2}{g}$.

The Greatest Height. The greatest height to which the projectile will rise is found by putting $x = \frac{v_0^2 \sin 2\,\alpha}{2\,g}$ in the equation of the curve and solving for y. This gives

$$h = \frac{v_0^2 \sin^2 \alpha}{2g}.$$

and the angle that gives the greatest height is $\alpha = 90°$. For this case $h = \frac{v_0^2}{2g}$. This is the case that has already been considered under the head of a body projected vertically upward.

91. Body projected up an Inclined Plane. — If the body is projected up an inclined plane, making an angle $\beta (\beta < \alpha)$ with the horizontal and passing through the point O (see Fig. 114), we desire to find the point at which the projectile will strike the plane. For any point in the plane we have $y = x \tan \beta$. The point where this plane cuts the parabolic path of the projectile is given by the equations

$$x_1 = \frac{2 v_0^2 \cos \alpha \sin (\alpha - \beta)}{g \cos \beta};$$

$$y_1 = \frac{2 v_0^2 \cos \alpha \tan \beta \sin (\alpha - \beta)}{g \cos \beta}.$$

The range on the plane

$$\frac{2 v_0^2 \cos \alpha \sin (\alpha - \beta)}{g \cos^2 \beta}.$$

Problem 133. The initial velocity v_0 is the same as that of a body falling freely from the directrix of the parabolic path to the point O on the curve. Show that the velocity of the body at any point on the curve is the same as would be acquired in falling freely from the directrix to that point.

Problem 134. A fire hose delivers water with a nozzle velocity v_0, at an angle of elevation α. How high up on a vertical wall, situated at a distance d' from the nozzle, will the water be thrown? It should be said that water thrown from a nozzle in a non-resisting medium takes a parabolic path and follows the same laws as projectiles.

Problem 135. What must be the nozzle velocity of water thrown upon a burning building, 200 ft. high, the angle of elevation of the curve being 60°?

Problem 136. The muzzle velocity of a gun is 500 ft. per second. Find its greatest range when stationed on the side of a hill which makes an angle of 10° with the horizontal: (*a*) up the hill, (*b*) down the hill. If the hillside is a plane, the area commanded by the gun is an ellipse, of which the gun is a focus.

Problem 137. A ball whose weight is 64.4 lb., shown in Fig. 115, starts from rest at A and rolls without friction in a circular path to the point B, where it is projected from the circular path horizontally. Find (*a*) the velocity at B, (*b*) the equation of its path after leaving B, and (*c*) the distance d from a vertical through B, where it strikes a horizontal 10 ft. below B.

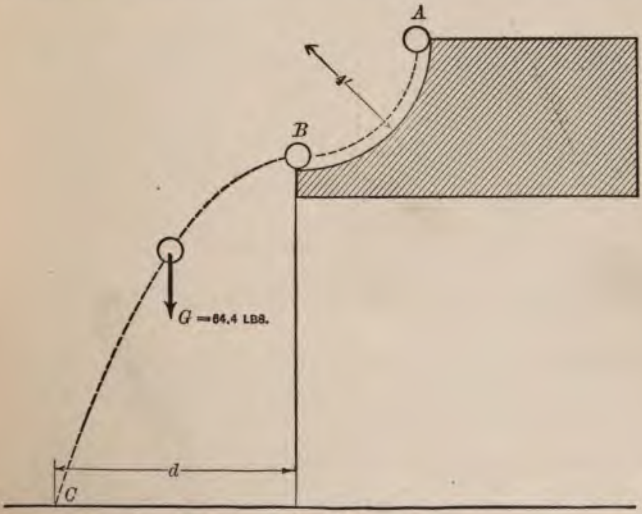


Fig. 115

Problem 138. If the body in Problem 137 had moved along a straight line from A to B and was then projected, find, as in the preceding problem, (*a*), (*b*), and (*c*).

Problem 139. A body whose weight is 12 lb. swings as a circular pendulum, as shown in Fig. 116, from A to B, when the string breaks. Find (*a*) the velocity at B, (*b*) the equation of its path after leaving B, and (*c*) the distance d where it strikes a horizontal 5 ft. below B.

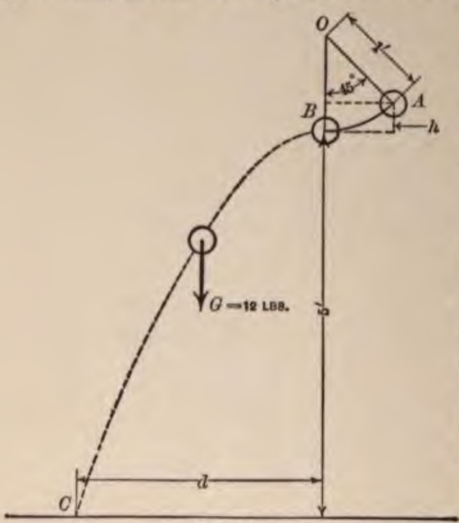


FIG. 116

Problem 140. A ball whose weight is 32.2 lb. starts from rest at A on the top of a sphere (Fig. 117), and rolls without friction to the point B, where it leaves the surface. Locate the point B. Find also (*a*) the angle of projection α, (*b*) the equation of the path of the body after leaving the sphere, and (*c*) the distance d where it strikes the horizontal.

Problem 141. The muzzle velocity of a gun situated at a height of 300 ft. above a horizontal plane is 2000 ft. per second. Find the area of plane covered by the gun.

Problem 142. The fly wheel shown in Fig. 84 "runs wild" and the rim breaks into six equal parts, free from the arms, when going at the speed of 300 revolutions per second. The path of the pieces being unimpeded, find the greatest height that could be reached by either piece and the greatest horizontal distance attainable.

92. Motion of Projectile in Resisting Medium. It was found by Rollins and others (see Encyclopædia Britannica

— "Gunnery") that the formula for projectiles in vacuo did not hold when the projectile moved in the atmosphere. That is, that the path followed by the projectile was not parabolic, but on account of the resistance of the atmosphere the range was much less than that given by the parabola.

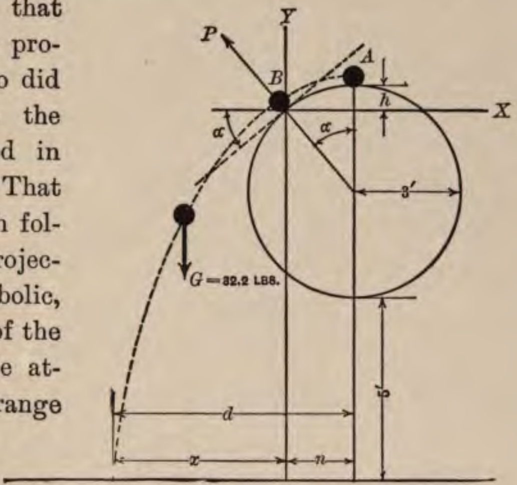


FIG. 117

A formula constructed by Hélie, empirically modifying the parabolic formula, is

$$y = x \tan \alpha - \frac{gx^2}{2 \cos^2 \alpha} \left(\frac{1}{v_0^2} + \frac{kx}{v_0} \right),$$

where $k = 0.0000000458 \ \frac{d^2}{w}$, d being the diameter of the projectile in inches, and w its weight in pounds. This gives the simplest formula for roughly constructing a range table.

Professor Bashforth of Woolrich found, from a series of experiments made by him, that for velocities between 900 and 1100 ft. per second the resistance varied as v^6, for velocities between 1100 and 1350 ft. per second the resistance varied as v^3, and for velocities above 1350 ft. per second the resistance varied as v^2.

In addition to the resistance of the air other factors tend to change the path of the projectile from the parabolic form, viz. the velocity of the wind and the rotation of the projectile itself. Most projectiles are given a right-handed rotation, and this causes them to bear away to the right upon leaving the gun. This is called *drift*. Correction is made for drift and wind velocity upon firing.

If the resistance of the air varies as the velocity, say it equals kv, then $kv_x = k\dfrac{dx}{dt}$ and $kv_y = k\dfrac{dy}{dt}$,

so that

(1) $$\frac{d^2x}{dt^2} = -k\frac{dx}{dt},\ \frac{d^2y}{dt^2} = -k\frac{dy}{dt} - g.$$

Integrating, and remembering that when $t = 0$,

$$v_x = v_0 \cos\alpha,\ v_y = v_0 \sin\alpha,$$

we have

(2) $$v_x = \frac{dx}{dt} = v_0 \cos\alpha \cdot e^{-kt},\ \ v_y = \frac{dy}{dt} = \frac{1}{k}[-g + (kv_0 \sin\alpha + g)e^{-kt}],$$

and therefore, since, when $t = 0$, $x = 0$, and $y = 0$,

(3) $$x = \frac{v_0}{k}\cos\alpha\,(1 - e^{-kt}),$$

(4) $$y = \frac{-g}{k}t + \frac{kv_0 \sin\alpha + g}{k^2}(1 - e^{-kt}).$$

Eliminating t, the equation of the curve is

(5) $$y = \frac{kv_0 \sin\alpha + g}{kv_0 \cos\alpha}\,x + \frac{g}{k^2}\log\frac{v_0\cos\alpha - kx}{v_0\cos\alpha}.$$

93. Path of Projectile Small Angle of Elevation. — When the resistance varies as the square of the velocity, the complete determination of the path of the projectile is mathematically difficult. In what follows, the angle of elevation has been assumed small so that powers of $\frac{dy}{dx}$ higher than the first have been neglected. Then $ds = dx$ and $s = x$. Let the resistance equal $kv^2 = k\left(\frac{ds}{dt}\right)^2$. Then

$$(1) \qquad \frac{d^2x}{dt^2} = -k\frac{ds}{dt}\frac{dx}{dt};$$

$$(2) \qquad \frac{d^2y}{dt^2} = -g - k\frac{ds}{dt}\frac{dy}{dt}.$$

Equation (1) may be put in the form

$$\frac{d\left(\frac{dx}{dt}\right)}{\frac{dx}{dt}} = -k\frac{ds}{dt},$$

which gives, upon integrating,

$$(3) \qquad \log\frac{\frac{dx}{dt}}{v_0 \cos\alpha} = -ks.$$

Since the initial value of $\frac{dx}{dt}$ is $v_0 \cos\alpha$, that is, when $t = 0$,

$$\frac{dx}{dt} = v_0 \cos\alpha.$$

Equation (3) may be written,

$$(4) \qquad \frac{dx}{dt} = v_x = v_0 \cos\alpha \cdot e^{-ks}.$$

Multiplying (1) by dy and (2) by dx and subtracting (2) from (1), we get

$$(5)\qquad \frac{d^2y\,dx - d^2x\,dy}{dt^2} = -g\,dx.$$

From (4) and (5) we have

$$\frac{d^2y\,dx - d^2x\,dy}{dx^2} = d\frac{dy}{dx} = -\frac{g}{v_0^2\cos^2\alpha}e^{2ks}dx,$$

and since $s = x$,

$$(6)\qquad d\frac{dy}{dx} = -\frac{g}{v_0^2\cos^2\alpha}e^{2kx}dx.$$

Integrating, and remembering that when $x = 0$, $\frac{dy}{dx} = \tan\alpha$, we have

$$(7)\qquad \frac{dy}{dx} - \tan\alpha = -\frac{g}{2\,kv_0^2\cos^2\alpha}(e^{2kx} - 1);$$

therefore,

$$(8)\quad y = x\tan\alpha + \frac{gx}{2\,kv_0^2\cos^2\alpha} - \frac{g}{4\,k^2v_0^2\cos^2\alpha}(e^{2kx} - 1).$$

If e^{2kx} be expanded in a series, this may be expressed approximately as follows,

$$(9)\qquad y = x\tan\alpha - \frac{gx^2}{2\,v_0^2\cos^2\alpha} - \frac{gkx^3}{3\,v_0^2\cos^2\alpha} \cdot \cdot \cdot$$

It is seen that if the third and following terms be neglected, the equation is that of the projectile in vacuo (Art. 90).

Problem 143. Find the range, greatest height, and time of flight, from Hélie's equation (Art. 92); Equation 5, Art. 92; and Equation 9 of the present article.

Problem 144. Compare the values obtained in the preceding problem with similar values obtained for the case of motion in vacuo, taking $\alpha = 45°$ and $v_0 = 1000$ ft. per second. In each case take $k = .0000000458\ \frac{d^2}{w}$, where $d = 6$ in. and $w = 150$ lb.

Problem 145. Find the angle of elevation α for each of the cases in preceding problem in order to strike a point 200 ft. high, distant 1000 ft. Take $v_0 = 1000$ ft. per second.

Problem 146. A locomotive weighing 175 tons moves in an 800 ft. curve with a velocity of 40 mi. per hour. Find the horizontal pressure on the rails, if they are on the same horizontal.

Problem 147. If the velocity of the earth was 18 times what it actually is, show that the force of gravity would not be sufficient to keep bodies on the earth near the equator. Take the radius of the earth as 4000 mi., and assuming the above conditions, find at what latitude the body would just remain on the earth.

Problem 148. The weight of a chandelier is 300 lb., and the distance of its center of gravity from the ceiling is 16 ft. Neglecting the weight of the supporting chain, find how much the tension in the chain will be increased if the chandelier is set swinging through an angle of 2°, measured at the ceiling.

Problem 149. A pail containing 5 lb. of water is caused to swing in a vertical circle at the end of a string 3 ft. long. Find the velocity of the pail at the highest point so that the water will remain in the pail. Find also the velocity of the pail at the lowest point.

94. Motion in Twisted Curve. — When the motion of a body is in a twisted curve, it is convenient to take account of its motion relative to three rectangular axes, x, y, and z. Let α, β, γ be the direction angles of the tangent line to the curve, and λ, μ, ν the direction angles of the resultant force. We may then write for the velocity,

$$v_x = v \cos \alpha = \frac{dx}{dt};$$

$$v_y = v \cos \beta = \frac{dy}{dt};$$

$$v_z = v \cos \gamma = \frac{dz}{dt};$$

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2},$$

and for the acceleration, since

$$\frac{dx}{dt} = \frac{dx}{ds}\frac{ds}{dt}, \quad \text{etc.}$$

$$a_x = a \cos \lambda = \frac{d^2x}{dt^2} = \frac{d^2s}{dt^2}\frac{dx}{ds} + \left(\frac{ds}{dt}\right)^2 \frac{d^2x}{ds^2};$$

$$a_y = a \cos \mu = \frac{d^2y}{dt^2} = \frac{d^2s}{dt^2}\frac{dy}{ds} + \left(\frac{ds}{dt}\right)^2 \frac{d^2y}{dt^2};$$

$$a_z = a \cos \nu = \frac{d^2z}{dt^2} = \frac{d^2s}{dt^2}\frac{dz}{dt} + \left(\frac{ds}{dt}\right)^2 \frac{d^2z}{dt^2}.$$

$$a = \sqrt{a_x^2 + a_y^2 + a_z^2} = \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2 + \left(\frac{d^2z}{dt^2}\right)^2},$$

since $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$ are the direction cosines of the tangent line and $\rho\frac{d^2x}{ds^2}$, $\rho\frac{d^2y}{ds^2}$, and $\rho\frac{d^2z}{ds^2}$ are the direction cosines of the principal normal.

From the above equations it will be seen that the resultant acceleration a may be resolved into tangential and normal components

$$a_t = \frac{dv}{dt} \text{ and } a_n = \frac{v^2}{\rho},$$

just as was done in the case of motion in a plane curve. In this case the normal is the principal normal and the radius ρ is the radius of absolute curvature.

As an illustration of motion in a twisted curve consider the motion in a helix. The helix may be considered as generated by the end of a line that moves with uniform velocity along the line OZ (Fig. 118). The edge of the thread of a screw is such a twisted curve. Let the curve

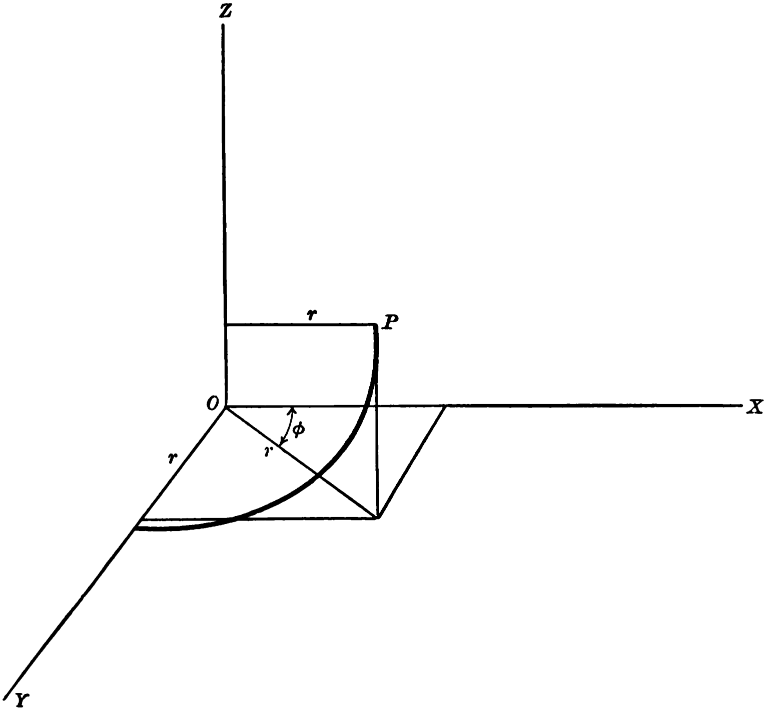


FIG. 118

be given by Fig. 118, and let P be any point having coördinates x, y, and z.

Then

$$x = r \cos \phi,$$

$$y = r \sin \phi,$$

$$z = \frac{k}{2\pi} \phi,$$

will represent the curve. It follows that

$$v_x = -r \sin\phi \frac{d\phi}{dt} = -r\omega \sin\phi\,;$$

$$v_y = r\cos\phi \frac{d\phi}{dt} = r\omega\cos\phi\,;$$

$$v_z = \frac{k}{2\pi}\frac{d\phi}{dt} = \frac{\omega k}{2\pi}.$$

$\frac{d\phi}{dt}$ is the angular velocity of the point with respect to z; represent it by ω (see Art. 95), so that $v = \omega\sqrt{r^2 + \frac{k^2}{4\pi^2}}$ = constant (k is an arbitrary constant that determines the pitch of the helix). The velocity of a point moving in such a curve is constant since ω is constant. The acceleration a_t is therefore zero.

We may also write

$$a_x = r\cos\phi \frac{d\phi}{dt} = -r\omega\cos\phi;$$

$$a_y = r\sin\phi \frac{d\phi}{dt} = -r\omega\sin\phi.$$

$$a_z = 0.$$

Therefore $a = \sqrt{a_x^2 + a_y^2 + a_z^2} = r\omega.$
That is, the acceleration in the direction of the resultant force is equal to ωr and the accelerating force is equal to $M\omega r$.

The fact of zero tangential accelerations has made this curve very useful. In many cases the helical surface formed by the revolving line has been made use of to send packages from upper floors of commercial establishments to the lower floors. Since $a_t = 0$, the packages move down with uniform motion. The helicoid is inclosed in a tube with convenient openings for the insertion of packages.

CHAPTER XI

ROTARY MOTION

95. Angular Velocity.—In Art. 71 linear velocity was defined as the rate of motion, and it was stated that it might be expressed as the ratio of distance to time or the rate of change of linear distance to time. The simplest case of rotating bodies is seen in uniform rotation about a fixed axis. The angular velocity is defined as the ratio of angular distance to time. Let the angular distance (measured in radians) be represented by α and the angular velocity by ω. Then for uniform velocity

$$\omega = \frac{\alpha}{t},$$

and for variable velocity

$$\omega = \frac{d\alpha}{dt}.$$

Angular velocity involves a magnitude and a direction, and may, therefore, be represented by an arrow (see Fig. 119), the length of the arrow representing the magnitude and drawn perpendicular to the plane of motion such that if you look along the arrow, from its point, the motion appears positive or negative; positive if counter-clockwise and negative if clockwise.

Velocity arrows may be compounded into a resultant or resolved into components in the same way that force arrows were treated. For example, in Fig. 119, the angular velocity of a body at any instant is represented by ω. Then the angular velocity with respect to two rectangular axes x and y will be represented by $\omega_x = \omega \cos \lambda$ and $\omega_y = \omega \sin \lambda$, so that $\omega^2 = \omega_x^2 + \omega_y^2$.

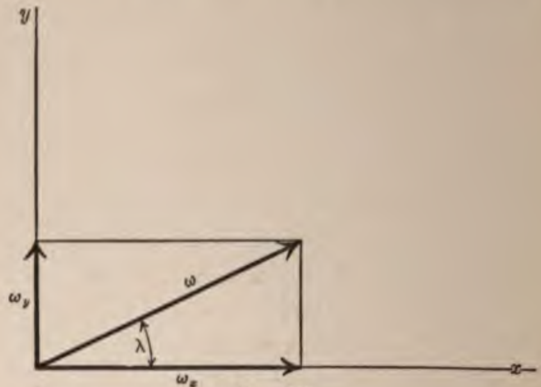


FIG. 119

In a similar way if a body has an angular velocity ω about an axis making angles λ, μ, and ν with the x, y, and z axes, respectively, the component velocities along the axes will be given by

$$\omega_x = \omega \cos \lambda,\ \omega_y = \omega \cos \mu,\ \omega_z = \omega \cos \nu;$$

so that

$$\omega^2 = \omega_x^2 + \omega_y^2 + \omega_z^2.$$

96. Angular Acceleration. — Angular acceleration, which we shall represent by θ, may be defined as the rate of change of angular velocity, so that if

$$\omega = \frac{d\alpha}{dt},\ \theta = \frac{d\omega}{dt} = \frac{d^2\alpha}{dt^2}.$$

From the preceding article and the definition of angular acceleration we may write

$$\theta_x = \frac{d\omega_x}{dt},\ \theta_y = \frac{d\omega_y}{dt},\ \theta_z = \frac{d\omega_z}{dt}.$$

The linear velocity and linear acceleration of a point of a rotating body may be determined in terms of the angular velocity and angular acceleration. Assume that for the instant under consideration the point is moving in the arc of a circle of radius ρ, over an arc $\alpha\rho$, then

$$v = \frac{\rho d\alpha}{dt} = \omega\rho \text{ and } a_t = \frac{\rho d^2\alpha}{dt^2} = \theta\rho.$$

It will be seen that v in this case is the velocity along a tangent to the path at the point P.

97. Angular Acceleration Constant. — In Art. 73 we found that when the linear acceleration was constant, the equations of motion reduced to a simple form. In a similar way if θ is constant, and the axis of rotation fixed, we have

$$\omega = \omega_0 + \theta t;$$

$$\alpha = \tfrac{1}{2}\theta t^2 + \omega_0 t;$$

$$\omega d\omega = \theta d\alpha;$$

$$\alpha = \frac{\omega^2 - \omega_0^2}{2\theta};$$

$$\alpha = \tfrac{1}{2}(\omega_0 + \omega)t,$$

where ω_0 is the constant initial angular velocity.

The expression for linear velocity and linear acceleration of any point P of the body becomes in this case $v = \omega r$ and $a_t = \theta r$, where r is the distance of P from the axis of rotation.

Problem 150. A fly wheel making 100 revolutions per minute is brought to rest in 2 min. Find the angular acceleration θ and the angular distance α passed over before coming to rest.

Problem 151. A fly wheel is at rest, and it is desired to bring it to a velocity of 300 radians per minute in ½ min. Find the acceleration θ necessary and the number of revolutions required. What is the velocity ω at the end of 10 sec.?

Problem 152. Suppose the fly wheel in Problem 151 to be 6 ft. in diameter. After arriving at the desired angular velocity, what is the tangential velocity of a point on the rim? What has been the tangential acceleration of this point, if constant?

98. Variable Acceleration. — In case θ is not constant, its law of variation must be given so that the equations of motion may be worked out. As an illustration suppose that a body moves in such a way that the angular acceleration θ varies as the angular distance α. Let $\theta = -k\alpha$, then from the equation $\omega d\omega = \theta d\alpha$, we get $\omega d\omega = -k\alpha d\alpha$. Taking the limits of ω as ω_0 and ω, and the limits of α, 0 and α, and of t, 0 and t, we have

$$\int_{\omega_0}^{\omega} \omega d\omega = -k\int_0^{\alpha} \alpha d\alpha;$$

therefore $$\omega = \sqrt{\omega_0^2 - k\alpha^2}.$$

Integrating again,

$$\int_0^{\alpha} \frac{d\alpha}{\sqrt{\omega_0^2 - k\alpha^2}} = \int_0 dt,$$

which gives

$$\frac{1}{\sqrt{k}} \sin^{-1} \frac{\sqrt{k}\alpha}{\omega_0} = t,$$

or $$\frac{\omega_0}{\sqrt{k}} \sin \sqrt{k}t = \alpha.$$

This last equation shows that α is a periodic function of the time; the motion is vibratory. Referring to Art. 79, it is seen that the motion is harmonic. In fact, if we substitute $\omega_0 = v_0\rho$ and $\alpha = s\rho$, we have exactly the same equation as was obtained in Art. 79. This example applies to the motion of a simple pendulum, considering it as rotating about the point of support.

Problem 153. The balance wheel of a watch goes backward and forward in $\frac{1}{2}$ sec. The angle through which it turns is 180°; find the greatest angular acceleration and the greatest angular velocity.

Problem 154. Assume the angular acceleration varies inversely as the square of the angular distance; find the relation between ω and α, and t and α.

99. Combined Rotation and Translation. — The angular acceleration of a body may be resolved into its tangential and normal components $a_t = \theta\rho$ and $a_n = \frac{v^2}{\rho} = \omega^2\rho$. If now the body has in addition to its rotation, a translation, the total acceleration of any point P will be given by the components $\theta\rho$, $\omega^2\rho$, and a_h. In Fig. 120 the body is supposed to have an axis of rotation perpendicular to the paper, and a translation parallel to ox. Let the angle that PO makes with x be β, so that $\cos\beta = \frac{x}{\rho}$ and $\sin\beta = \frac{y}{\rho}$.

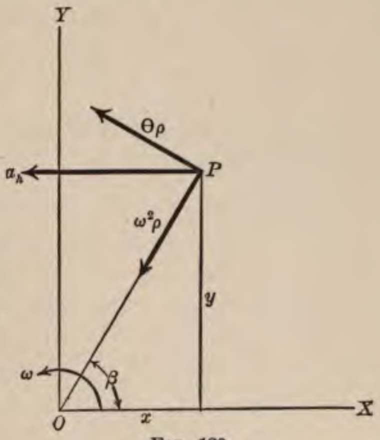


FIG. 120

Writing the x and y components of the acceleration, we have

$$a_x = -\omega^2 x - \theta y - a_h,$$
$$a_y = -\omega^2 y + \theta x.$$

The *tangential* and *normal* components of a are, from Fig. 120,

$$a_t = \theta\rho + a_h\left(\frac{y}{\rho}\right), \quad a_n = \omega^2\rho + a_h\left(\frac{x}{\rho}\right).$$

As an illustration of combined rotation and translation consider the case of a wheel of radius r rolling in a straight horizontal track. Let the acceleration of translation of the center be a_h, and the angular velocity of the wheel, about the center, be ω, and the corresponding angular acceleration θ.

Then

$$a_t = 2\,\theta r + a_h,$$
$$a_n = 2\,\omega^2 r,$$

are the tangential and normal accelerations of a point on the rim situated at the top of the wheel.

Problem 155. A locomotive drive wheel 6 ft. in diameter rolls along a level track. Find the greatest tangential acceleration and the greatest normal acceleration of any point on the tread, (*a*) when the velocity v with which the wheel moves along the track is 60 mi. per hour, (*b*) when the engine is slowing down uniformly and has a velocity of 30 mi. per hour at the end of 3 min., (*c*) when the engine is starting up uniformly and has a velocity of 30 mi. per hour at the end of 5 min.

Problem 156. A cylinder, diameter d, rolls from rest down an inclined plane, inclined at an angle ϕ with the horizontal. What is the greatest normal and greatest tangential acceleration of any point on its circumference? Neglect friction.

100. Rotation in General.—It has been shown in Art. 36 that any system of forces acting upon a rigid body may be reduced to a single force and a single couple whose plane is perpendicular to the line of action of the single force. That is, the most complicated cases of rotation consist of an instantaneous translation combined with an instantaneous rotation at right angles to the translation.

Bodies projected into the air while rotating have been mentioned in Art. 92. The projectile rotating about an axis is projected in the direction of the axis. If no forces acted upon it after leaving the gun, it could move in a straight line. It is, however, acted upon by gravity, which causes it to take a somewhat parabolic path. The resistance of the air causes the projectile to drift.

This action of the projectile will probably be most easily explained by a consideration of the motion of a baseball. The modern pitcher when he throws the ball gives it also a motion of rotation. The force of gravity causes the ball to take a path somewhat parabolic and the resistance of the air, due to the rotation, causes the ball to de-

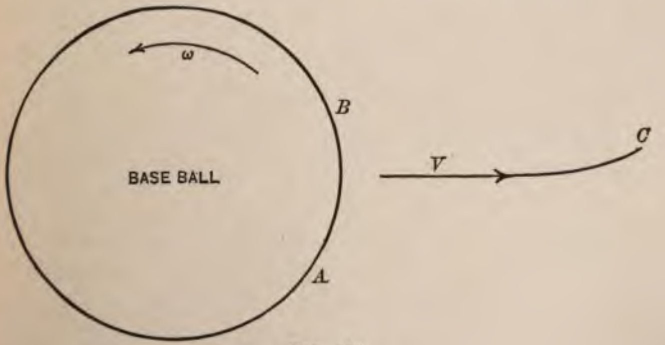


FIG. 121

flect from the plane in which it started. The combination of the two deflecting forces makes the path of the ball a twisted curve. Different speeds and directions of rotation and different speeds of translation give great variety to the curves produced. The action of the baseball will be best understood by referring to Fig. 121. Let the baseball have an initial angular velocity ω and an initial linear velocity v in the directions shown. The rotation of the ball causes the air to be more dense at A than at B, so that the ball is pushed constantly from A to B. This action causes it to deviate from the plane in which it initially moved and to take the path indicated by c. As stated above, this action in the case of a projectile is known as drift.

CHAPTER XII

DYNAMICS OF MACHINERY

101. Statement of D'Alembert's Principle. — A body may be considered as made up of a collection of individual particles held together by forces acting between them. The motion of a body concerns the motion of its individual particles. We have seen that in dealing with such problems as the motion of a pendulum it was necessary to consider the body as concentrated at its center of gravity; that is, to consider it as a material point. The principle due to D'Alembert makes the consideration of the motion of bodies an easy matter. Consider a body in motion due to the application of certain external forces or *impressed forces.* Instead of thinking of the motion as being produced by such impressed forces, imagine the body divided into its individual particles and imagine each of the particles acted upon by such a force as would give it the same motion it has due to the impressed forces. These forces acting upon the individual particles are called the *effective forces.* D'Alembert's Principle, then, states *that the impressed forces will be in equilibrium with the reversed effective forces.*

It must be seen by the student that the principle does not deal with the forces acting between the particles of a body; these are considered as being in equilibrium among themselves. We shall see in what follows that this principle, by assuming a system of effective forces acting upon the particle, enables us in many cases to apply the principles of equilibrium as developed and used in the subject of statics.

102. Simple Translation of a Rigid Body. — The principle of D'Alembert will be best understood by applying it to the consideration of the simpler motions of a rigid body. Let us consider the body, Fig. 121 a, and let us assume that it has simple translation parallel to x due to the action of certain impressed forces P_1, P_2, P_3, etc., making angles α_1, α_2, α_3, etc., with x. It is seen at once that only the components of P_1, P_2, P_3, etc., parallel to x have any part in producing motion in that direction. We may say, then, that the impressed forces are $P_1 \cos \alpha_1$, $P_2 \cos \alpha_2$, $P_3 \cos \alpha_3$, etc., and that these produce an acceleration a in the direction indicated.

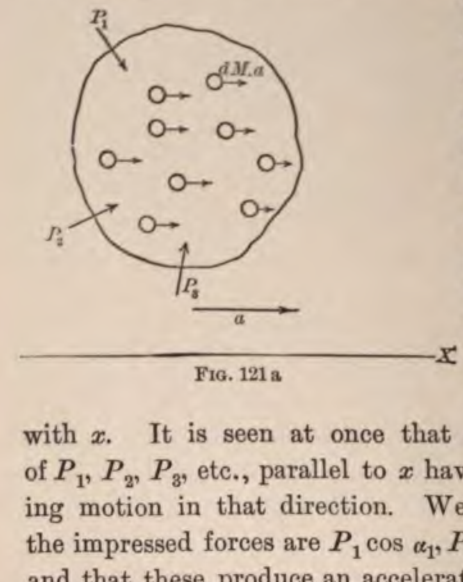


FIG. 121 a

Imagine the body now divided into small particles each of mass dM, and assume that the system of forces producing the motion of the body consists of a small force $dM.a$ acting on each particle. D'Alembert's Principle then states that these forces reversed are in equilibrium with the impressed forces. We have, then, $\Sigma x = 0$, or $\quad P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + P_3 \cos \alpha_3 + \text{etc.} - \Sigma dM \cdot a = 0,$

or $\quad P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + P_3 \cos \alpha_3 + \text{etc.} = \Sigma dM \cdot a.$

But since motion is parallel to x, it is evident that

$P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + P_3 \cos \alpha_3 + \text{etc.} = \text{Resultant Force} = R.$

Therefore, for continuous bodies,

$$R = a \int dM = aM,$$

since a is the same for every particle of the body. Consider each particle at a distance y from x and let d be the distance of R from x; then taking moments with respect to an axis through x and perpendicular to it, we have

$$Rd = a \int y dM = a \bar{y} M,$$

where $\bar{y}$ is the distance of the center of gravity of the body from x (Art. 22). Dividing through by R, we find,

$$d = \bar{y};$$

that is, the resultant force passes through the center of gravity of the body.

103. Simple Rotation of a Rigid Body. — We shall now apply D'Alembert's Principle to the case of a rigid body rotating about a fixed axis. Let B in Fig. 122 be the body, and imagine it rotating in the direction indicated about an axis through O perpendicular to the paper. Suppose the rotation due to the action of forces P_1, P_2, P_3, P_4, etc., making angles α_1, β_1, γ_1; α_2, β_2, γ_2, etc., with a set of axes x, y, z, with origin at O. It is evident that only the components of the forces P_1, P_2, P_3, P_4, etc., parallel to the xz-plane, will have any part in producing rotation. Call these projections P_1', P_2', P_3', P_4', etc.; they are the impressed forces for the motion considered. The distances of the lines of action of these forces from O may be represented by d_1, d_2, d_3, d_4, etc.

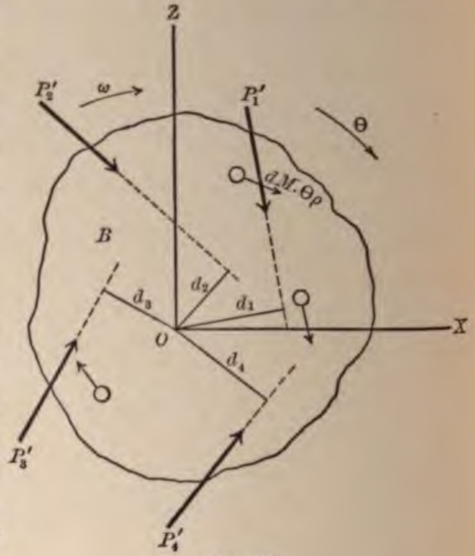


FIG. 122

Now consider the effective forces. Imagine the body made up of individual particles dM situated at a distance ρ from O. Each dM is acted upon by a force $dMa_t = dM\theta\rho$. These are the effective forces. Equating the moments of these forces, reversed, to the moments of the impressed forces, we have

$$\Sigma\,(P_1'd_1 + P_2'd_2 + P_3'd_3 + \text{etc.}) = \int dM\cdot\theta\rho\cdot\rho = \theta\int\rho^2 dM = \theta I,$$

since $\int\rho^2 dM$ gives the moment of inertia of B with respect to O. (See Art. 37.)

That is, *when a body rotates about a fixed axis, the sum of the moments of the impressed forces in the plane of rotation equals θI.*

It is evident that any one of the forces P_1', P_2', P_3', P_4', etc., may be such as to offer a resistance to the indicated motion of the body. In such a case the sign of its moment would be changed.

104. Reactions of Supports; Rotating Body. — It has just been shown that one equation is sufficient to give the motion of a rigid body about a fixed axis. It is necessary, however, in order to determine the reactions of the supports, to use other equations. Consider the body B with its axis vertical, as shown in Fig. 123, and let the rotation take place as indicated due to the action of the forces P_1, P_2, P_3, P_4, etc. Let P_x, P_y, and P_z be the reaction of the supports on the axis at O, and P'_x and P'_y the reactions of the support at A on the axis.

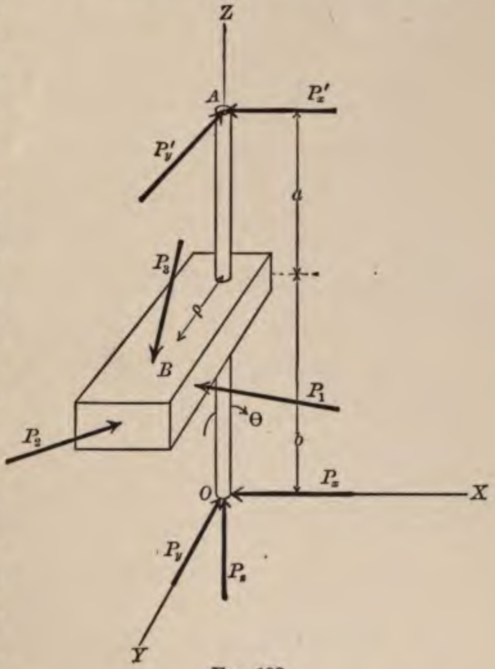


FIG. 123

The effective forces acting upon a particle of the body B are shown in Fig. 124. Let dM be its mass and replace the resultant force acting on dM by its tangential and normal components and call them dT and dN respectively. It has been shown (Art. 86) that the normal force equals $\frac{Mv^2}{\rho}$ and the tangential force equals Ma_t, and that $v = \omega\rho$ and $a_t = \theta\rho$ (Art. 96). We have then

$$dN = dM\omega^2\rho \text{ and } dT = dM\theta\rho,$$

and it is seen that both dN and dT may be resolved along each of the axes x, y, and z for every dM of the body.

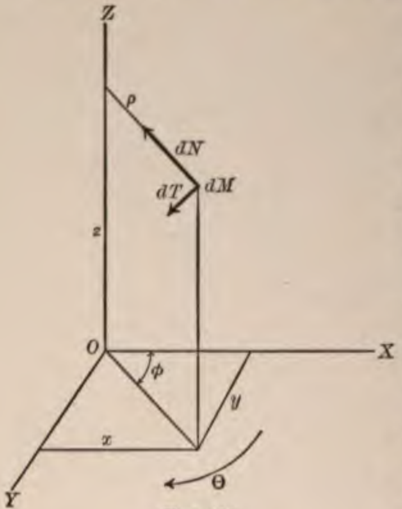


FIG. 124

From what has been said it is evident that the sum of the impressed forces along the x-axis equals the sum of the reversed effective forces along the same axis. Calling the impressed forces i and the effective forces e, we may write

$$\begin{aligned} \Sigma x_i &= \Sigma x_e, & \Sigma \text{ mom}_{ix} &= \Sigma \text{ mom}_{ex}, \\ \Sigma y_i &= \Sigma y_e, & \Sigma \text{ mom}_{iy} &= \Sigma \text{ mom}_{ey}, \\ \Sigma z_i &= \Sigma z_e; & \Sigma \text{ mom}_{iz} &= \Sigma \text{ mom}_{ez}. \end{aligned}$$

That is, the components of the impressed forces along each of the three axes x, y, and z equal the components of the reversed effective forces along these axes and the moment of the impressed forces with respect to the three axes x, y, and z equals the moment of the reversed effective forces with respect to these axes.

Writing down these six conditions, we have

$$\Sigma x_i = -\int dN \cos\phi + \int dT \sin\phi,$$

$$\Sigma y_i = -\int dN \sin\phi - \int dT \cos\phi,$$

$$\Sigma z_i = 0;$$

$$\Sigma \operatorname{mom}_{ix} = -\int dN \sin\phi \cdot z + \int dT \cos\phi \cdot z,$$

$$\Sigma \operatorname{mom}_{iy} = \int dN \cos\phi \cdot z + \int dT \sin\phi \cdot z,$$

$$\Sigma \operatorname{mom}_{iz} = \int dT \cdot \rho.$$

Substituting the values of dN and dT in the expressions on the right-hand side, we have

$$\int dN \cos\phi = \omega^2 \int \rho \cos\phi dM = \omega^2 \int x dM = -\omega^2 M\bar{x},$$

$$\int dN \sin\phi = \omega^2 \int y dM = -\omega^2 M\bar{y},$$

$$\int dN \sin\phi \cdot z = \omega^2 \int yz dM,$$

$$\int dN \cos\phi \cdot z = \omega^2 \int xz dM,$$

$$\int dT \sin\phi = \int \theta\rho dM \sin\phi = \theta \int y dM = \theta M\bar{y},$$

$$\int dT \cos\phi = \theta \int x dM = \theta M\bar{x},$$

$$\int dT \sin\phi \cdot z = \theta \int yz dM,$$

$$\int dT \cos\phi \cdot z = \theta \int xz dM.$$

The six general equations therefore reduce to the form

$$\Sigma x_i = \theta M\bar{y} - \omega^2 M\bar{x}, \tag{1}$$

$$\Sigma y_i = -\theta M\bar{x} - \omega^2 M\bar{y}, \tag{2}$$

$$\Sigma z_i = 0; \tag{3}$$

$$\Sigma \operatorname{mom}_{ix} = -\omega^2 \int yz\,dM + \theta \int xz\,dM, \tag{4}$$

$$\Sigma \operatorname{mom}_{iy} = \omega^2 \int xz\,dM + \theta \int yz\,dM, \tag{5}$$

$$\Sigma \operatorname{mom}_{iz} = \theta \int \rho^2 dM = \theta I_z. \tag{6}$$

These equations hold true at any instant during the motion of the body. It will be seen, since $\bar{x}$ and $\bar{y}$ are the coördinates of the center of gravity of the body, that when the axis of rotation passes through the center of gravity, the right-hand sides of (1) and (2) become zero. It is further seen from (4) and (5) that if the body B has a plane of symmetry as the xy-plane, the right-hand sides of these equations reduce to zero, since for every $\int x(+z)dM$ there is a corresponding $\int x(-z)dM$ and for every $\int y(+z)dM$ there is a corresponding $\int y(-z)dM$. Therefore, *when the axis of rotation passes through the center of gravity and the plane xy is a plane of symmetry, the six equations become:*

$$\Sigma x_i = 0, \tag{7}$$

$$\Sigma y_i = 0, \tag{8}$$

$$\Sigma z_i = 0; \tag{9}$$

$$\Sigma \operatorname{mom}_{ix} = 0, \tag{10}$$

$$\Sigma \operatorname{mom}_{iy} = 0, \tag{11}$$

$$\Sigma \operatorname{mom}_{iz} = \theta I_z. \tag{12}$$

s case is the one that usually comes up in engineer- oblems, and so these simplified equations are more used than the six more general equations. It will ticed that these equations are exactly the same as nditions for equilibrium as determined in Art. 35 t that Σ mom_{iz} is not zero.

Rotation of a Sphere. — Suppose the body *B*, Fig. o be a cast-iron , radius 2 in., con- l to the axis by a tless arm whose is 6 in. Let the be rotated by a running over a of radius 1 in. ed 2 in. below P'_x. he constant ten- n the cord 10 lb. uppose it acts in -plane. Suppose ft. and $b = 6$ in. ider the motion the sphere is in z-plane. Take y-plane through nter of the sphere ndicular to z, then M and $\int yzdM$ are both zero. Using the foot-pound-l system of units, we have, $\bar{x} = \bar{y} = \frac{1}{2}$, $M = \frac{1}{4}$, $I_z =$ $G = 8.05$ lb., and

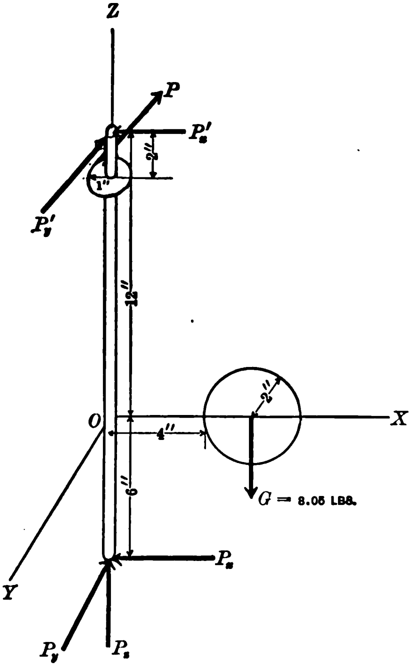


Fig. 125

$$\Sigma x_i = P'_x + P_x = \frac{\theta}{8} - \frac{\omega^2}{8}, \qquad (a)$$

$$\Sigma y_i = -10 - P'_y - P_y = -\frac{\theta}{8} - \frac{\omega^2}{8}, \qquad (b)$$

$$\Sigma z_i = P_z - 8.05 = 0; \qquad (c)$$

$$\Sigma \text{ mom}_{ix} = -P'_y - 10(\tfrac{5}{6}) + P_y(\tfrac{1}{2}) = 0, \qquad (d)$$

$$\Sigma \text{ mom}_{iy} = P'_x - (8.05)(\tfrac{1}{2}) - P_x \tfrac{1}{2} = 0, \qquad (e)$$

$$\Sigma \text{ mom}_{iz} = \tfrac{10}{12} = \theta(.065). \qquad (f)$$

From equation (*f*) we get $\theta = 12.81$ radians per (second)^2. Suppose the body begins to rotate from rest and that at the time under consideration it has been rotating one second. From the relation $\omega = \omega_0 + \theta t$ (Art. 97) we get $\omega = \theta = 12.81$ radians per second. Solving the remaining equations, we get $P'_x = -3.623$ lb.; $P'_y = -1.507$ lb.; $P_z = 8.05$ lb.; $P_x = -15.284$ lb.; $P_y = 13.616$ lb. The negative signs indicate that the arrows in the figure have been assumed in the wrong direction.

Problem 157. A fly wheel 3 ft. in diameter rotates about a vertical axis. The cross section of the rim is 3 in. × 3 in. and is made of cast-iron. Neglect the weight of the spokes. This wheel is placed on the axis in the preceding problem instead of the sphere. If the other conditions are the same, find the reactions of the supports.

Problem 158. The sphere in Fig. 125 is replaced by a right circular cast-iron cone of height one foot and diameter of base one foot. The vertex is placed at the point of attachment of the sphere. If the other conditions are the same, find the reactions of the supports.

Problem 159. In the problem of the sphere, Art. 105, find the angular velocity at the end of 30 sec. and the reactions of the supports for such speed.

106. Center of Percussion. — It will be interesting now to consider the motion of a slender homogeneous rod due to an impulsive force P when free to swing about a horizontal axis through one end. Let the rod be given in Fig. 126 and let O be the axis perpendicular to the paper about which rotation takes place. The length of the rod is l, and P is the impressed force tending to produce rotation.

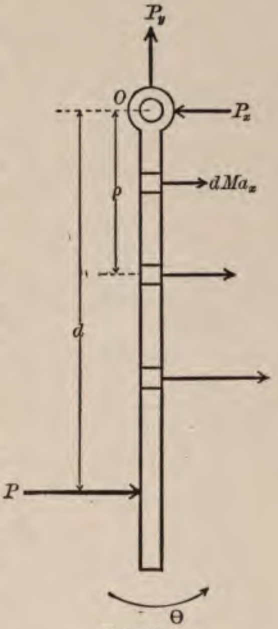


FIG. 126

The effective forces are represented in the figure as acting on each individual particle, equal in each case to $dM \cdot a_x = dM\theta\rho$, where ρ is the distance of dM from O. D'Alembert's Principle for the horizontal forces gives

$$P - P_x = \int dM\theta\rho = \theta\int\rho dM = \theta\bar{\rho}M;$$

similarly moments about the axis through O,

$$Pd = \theta\int\rho^2 dM = \theta I_z.$$

But I_z for a slender rod has been shown to be

$$\frac{1}{3}Ml^2 \text{ and } \bar{\rho} = \frac{l}{2},$$

so that $P_x = P - \theta M\bar{\rho} = P - \frac{Pd}{I_z}M\bar{\rho} = P\left(1 - \frac{3}{2}\frac{d}{l}\right).$

If $P_x = 0$, then $d = \frac{2}{3}l$. Under such conditions, if the rod were struck with a blow P, there would be no horizontal reaction at O. This point distant $\frac{2}{3}l$ from O, for which $P_x = 0$, is called the *center of percussion*. The general problem of center of percussion is not quite within the scope of the present work.

A right circular cylinder of height h and diameter of base d is made of cast iron. Locate its center of percussion, when supported as the rod in Fig. 126.

107. Compound Pendulum. — When a body rotates about a horizontal axis due to the action of gravity, it is called a compound pendulum. We have seen how to find the time of vibration of a simple pendulum and can investigate its motion completely, for small oscillations. We shall now study the motion of the compound pendulum.

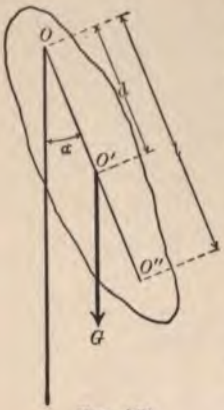


Fig. 127

Let the pendulum be represented by Fig. 127 and suppose the axis of rotation is through O perpendicular to the paper. Taking moments about the axis of rotation, we have

$$-Gd\sin\alpha = \theta I_0,$$

$$\theta = -\frac{Gd\sin\alpha}{I_0} = -\frac{Gd\sin\alpha}{Mk_0^2} = -\frac{gd\sin\alpha}{k_0^2}.$$

It is seen that θ varies with $\sin\alpha$. We wish now to find the length of a simple pendulum that will have the same period of vibration as this compound pendulum.

It was found, Art. 88, that the tangential acceleration for a simple pendulum, $a_t = -g \sin \alpha$, and hence its angular acceleration $= \frac{-g \sin \alpha}{l}$. Equating this value to the value of θ found for the compound pendulum, we get

$$l = \frac{k_0^2}{d},$$

as the length of a simple pendulum having the same period of vibration. This length l is the length of the compound pendulum. That is, the length of a compound pendulum is the length of a simple pendulum having the same period of vibration.

The student may find this length experimentally by taking a piece of thread with a small lead ball attached to one end, and holding the other end at O adjust the length of the thread until the compound pendulum and the simple pendulum vibrate simultaneously. The length of the thread is the length l.

Since $l = \frac{k_0^2}{d}$, and $k_0^2 = k_{0'}^2 + d^2$,

we may write

$$d(l - d) = k_{0'}^2, \text{ or } OO' \cdot O'O'' = \text{constant}.$$

Evidently the relation is not changed if O' and O'' be interchanged; we may, therefore, say that the compound pendulum will vibrate with the same period when suspended about O'' as an axis. This point is called the *center of oscillation*. The point O is known as the *point of suspension*. The result may be stated as follows: *in a* *compound pendulum the point of suspension and the center of oscillation are interchangeable.*

The time of vibration of a simple pendulum was found to be $t = \pi\sqrt{\frac{l}{g}}$ (Art. 88). This gives for the time of vibration of a compound pendulum,

$$t = \pi\sqrt{\frac{k_0^2}{gd}}.$$

Problem 160. A cast-iron sphere whose radius is 6 in. vibrates as a pendulum about a tangent line as an axis. Find the period of vibration and the length of a simple pendulum having the same period. Locate the center of oscillation.

Problem 161. A steel rod one inch in diameter and 3 ft. long is free to turn about a horizontal axis through one end. While hanging from this axis it is suddenly acted upon by a 10-lb. force perpendicular to its length in such a way as to cause the horizontal component of the reaction of the support to be zero. How far from the support does the force act?

Problem 162. Two drums whose radii are $r_1 = 16$ in. and $r_2 = 12$ in. are mounted as shown in Fig. 128. Their combined weight is 200 lb. and $k = 14''$. The forces G_1 and G_2 act upon the drum, as well as journal friction amounting to 16 lb. The radius of the shaft is 1 in. Find the velocities of G_1 and G_2 and the drums when the point of attachment of the cord on

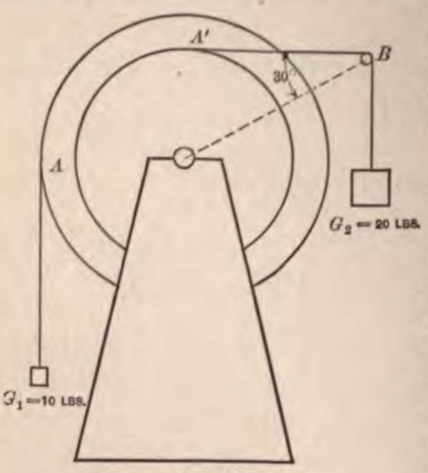


FIG. 128.

the small drum has traveled from rest at A to a point A'. Neglect the friction at B.

108. Experimental Determination of Moment of Inertia.—The computation of the moment of inertia of many bodies is a difficult matter. It is often convenient, therefore, to use an experimental method in dealing with such bodies. The compound pendulum furnishes a means whereby such determinations may be made. From Art. 107, we find that the time of vibration of a compound pendulum is

$$t = \pi \sqrt{\frac{k_0^2}{gd}}.$$

This may be written

$$k_0^2 = \frac{t^2}{\pi^2} gd\,;$$

multiplying both sides by M, the mass of the body, we have

$$I_0 = M\frac{t^2}{\pi^2} gd = G\frac{t^2}{\pi^2}\,d.$$

It thus appears that if d, the distance from O to the center of gravity, is known (the center of gravity may be located by balancing over a knife edge) and also the weight G, and the body be allowed to swing as a pendulum about O as an axis, t may be determined, giving I_0.

If $I_{0'}$ be desired, it may be determined from the formula (see Art. 41),

$$I_{0'} = I_0 - Md^2.$$

Problem 163. The connecting rod of a high-speed engine tapers regularly from the cross-head end to the crank-pin end. Its length is 10 ft., its cross section at the large end 5.59″ × 12.58″ and at the cross-head end 5.59″ × 8.39″. Neglecting the holes at the ends, the center of gravity is 64 in. from the cross-head end. The rod is made of steel and vibrates as a pendulum about the cross-head end in 1.3 sec. Compute its moment of inertia.

Problem 164. The student should take such a connecting rod as the one in the preceding problem and by swinging it as a pendulum find its period of vibration. Compute the moment of inertia.

109. Determination of g.—From the preceding article we see that

$$g=\frac{k^2{}_0\pi^2}{dt^2}=\frac{I_0\pi^2}{Mt^2d},$$

this relation enables us to determine g, as soon as we know I_0, M, and d, by determining the time of vibration about the point O. It is evident that $\frac{I_0\pi^2}{Md}$ is a constant for the body, when the axis is through O, and that when once determined accurately the pendulum might be used to determine g for any locality.

This constant, $\frac{I_0\pi^2}{Md^2}$, is known as the *pendulum constant.*

Problem 165. A round rod of steel 6 ft. long is made to swing as a pendulum about an axis tangent to one end and perpendicular to its length. The rod is 1 in. in diameter. Determine the pendulum constant.

Problem 166. The center of gravity of a connecting rod 5 ft. long is 3 ft. from the cross-head end. The rod is vibrated as a pendulum about the cross-head end. It is found that 50 vibrations are made in a minute. Find the radius of gyration with respect to the cross-head end.

110. The Torsion Balance.—A torsion balance consists of a body such as ABC, Fig. 129, suspended by means of a slender rod or wire rigidly clamped at both ends. Suppose the wire clamped at O, and let the body ABC

be a cast-iron disk of radius r and thickness t. The point O of support is the center of gravity. The mark OA on the body is shown in the neutral position. The application of a certain torque in the plane of the disk causes it to turn through a certain angular distance so that OA assumes periodically the positions OB and OC, due to the resistance of the wire. It is well known that a circular rod or wire when twisted offers a resistance to the twist, such that the resisting torque varies as the angle of displacement. As the line OA moves to OB the resisting torque offered by the wire steadily increases. After the body has been given a twist the only forces tending to produce rotation are the forces in the wire. So that if we call the moment of the couple m we may write $m = c\alpha$, since the moment of resistance varies with α. If, for the particular wire in question, when $\alpha = \alpha_1$ that $m = m_1$ we may write, since c is constant, $c = \frac{m_1}{\alpha_1}$.

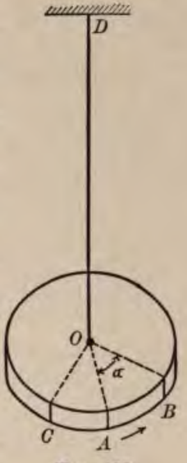


FIG. 129

Taking moments about the axis of rotation, we have

$$m = \theta I_z,$$

or

$$-\frac{m_1}{\alpha_1}\alpha = I_z\frac{\omega d\omega}{d\alpha},$$

since $\omega d\omega = \theta d\alpha$, and the resisting torque is negative. Multiplying through by $d\alpha$ and integrating, we get, if $\omega = 0$ when $\alpha = \alpha_0$,

$$-\frac{m_1}{\alpha_1}\left(\frac{\alpha^2}{2}-\frac{\alpha_0^2}{2}\right)=I_z\frac{\omega^2}{2},$$

or $$\omega=\sqrt{\frac{m_1}{I_z\alpha_1}}\sqrt{\alpha_0^2-\alpha^2}.$$

But $$\omega=\frac{d\alpha}{dt}, \text{ so that}$$

$$dt=\sqrt{\frac{I_z\alpha_1}{m_1}}\,\frac{d\alpha}{\sqrt{\alpha_0^2-\alpha^2}}.$$

If we integrate, taking $t=0$, when $\alpha=\alpha_0$, we have

$$t=\left[\sqrt{\frac{I_z\alpha_1}{m_1}}\sin^{-1}\frac{\alpha}{\alpha_0}\right]_{\alpha_0}^{\alpha}=\sqrt{\frac{I_z\alpha_1}{m_1}}\left(\sin^{-1}\frac{\alpha}{\alpha_0}-\frac{\pi}{2}\right),$$

where t is the time taken in turning from OB through any angle α. Suppose the upper limit zero ($\alpha=0$), then

$$\sin^{-1}\frac{\alpha}{\alpha_0}=0,\ \pi, \text{ etc.}$$

Suppose $$\sin^{-1}\frac{\alpha}{\alpha_0}=\pi,$$

then $$t=\frac{\pi}{2}\sqrt{\frac{I\alpha_1}{m_1}}.$$

The value of t given represents one fourth of a complete backward and forward swing, so that for a complete period

$$t=2\pi\sqrt{\frac{I\alpha_1}{m_1}}.$$

It is seen that this time of vibration is independent of the initial angular displacement α_0.

It is also seen that the moment of inertia of a body might be determined by suspending it as the body ABC is suspended. The constant $c = \frac{m_1}{a_1}$ is a constant of the wire or rod and depends upon the material and diameter. Knowing this constant, it would only be necessary to determine the period of vibration in order to find I.

For practical purposes, however, it is desirable to eliminate from consideration the value $\frac{m_1}{a_1}$. For this purpose suppose the disk provided with a suspended platform rigidly attached as shown in cross section in Fig. 130. Let t be its time of vibration and I its moment of inertia about the axis of suspension. Now place on the disk two equal cylinders H in such a way that their center of gravity is the axis of suspension. Let t_1 be the period of vibration of the cylinders and support and I_1 their moment of inertia.

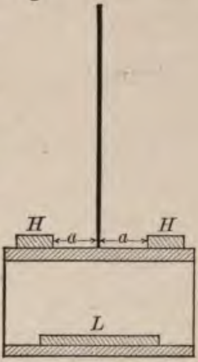


FIG. 130

Then $\frac{t^2}{t_1^2} = \frac{I}{I_1}$. The moment of inertia of the two cylinders with respect to the axis of rotation is known; call it I_2. Then $I_1 = I + I_2$,

so that
$$I = I_2 \frac{t^2}{t_1^2 - t^2}.$$

This gives the moment of inertia of the torsion balance, which, of course, is a constant.

The moment of inertia of any body L may now be determined by placing the body on the suspended platform with its center of gravity in the axis of rotation and noting the time of vibration. Calling the time of vibration of the body L and the balance t_1 and their moment of inertia I_1, we have

$$\frac{t^2}{t_1^2}=\frac{I}{I_1}$$

Let the moment of inertia of L itself be I_L, so that

$$I_1=I+I_L$$

Then

$$I_L=I\frac{t_1^2-t^2}{t^2}$$

This method may be used in finding the moment of inertia of non-homogeneous bodies, provided the center of gravity be placed in the axis of rotation.

Problem 167. The moment of inertia of a torsion balance is 6300 and its time of vibration 20 sec. The body L consists of a homogeneous cast-iron disk 3 in. in diameter and 1 in. thick. Find the time of vibration of the balance when L is in place. Assume the moment of inertia of the disk.

Problem 168. The same balance as that used in the preceding problem is loaded with a body L, and the time of vibration is found to be 30 sec. Determine the moment of inertia of L.

111. Constant Angular Velocity. — The six general equations may be written,

$$P_x'+P_x+\Sigma x=\theta M\bar{y}-\omega^2M\bar{x},$$
$$P_y'+P_y+\Sigma y=-\theta M\bar{x}-\omega^2M\bar{y},$$
$$P_z'+P_z+\Sigma z=0,$$

$$P_y'a + P_yb + \Sigma(yZ - zY) = -\omega^2\int yzdM + \theta\int xzdM,$$

$$P_x'a + P_xb + \Sigma(zX - xZ) = \omega^2\int xzdM + \theta\int yzdM,$$

$$\Sigma(xY - yX) = \theta I_z,$$

where the P's represent the action on the bearings and ΣX, ΣY, and ΣZ represent the components of the forces producing the rotation and $\Sigma(yZ - zY)$, etc., the moments of these forces. Now if X, Y, and Z are each zero, the last equation shows that $\theta = 0$, and therefore the angular velocity ω is constant. The condition, however, that X, Y, and Z be each equal zero, means that the forces tending to produce rotation no longer exist. Such an axis is sometimes called a permanent axis. It is a principal axis of the body for the point.

112. Rigid Body Free to Rotate. — If in addition to the conditions that X, Y, Z, $\int xzdM$ and $\int yzdM$ be each zero, we impose the condition that both $\bar{x}$ and $\bar{y}$ be zero, that is, that the z-axis pass through the center of gravity, the equations of motion become

$$\begin{aligned} P_x' + P_x &= 0, \\ P_y' + P_y &= 0, \\ P_z' + P_z &= 0, \\ P_y'a + P_yb &= 0, \\ P_x'a + P_xb &= 0, \\ \theta &= 0. \end{aligned}$$

The body is in equilibrium under the action of the reactions of the supports and continues to rotate with uniform velocity ω about the original axis of rotation.

The axis of rotation is now a principal axis of the body through the center of gravity. It is often called an axis of free rotation. Since there are three principal axes of the body through the center of gravity, there are three free axes of rotation.

113. Rotation of Symmetrical Bodies. — When a homogeneous body having a plane of symmetry rotates with constant angular velocity ω about an axis perpendicular to that plane, the only forces acting on the body reduce to

$$P = \omega^2 M \bar{\rho},$$

where P is the centripetal force acting through the center of gravity and $\bar{\rho}$ is the distance of the center of gravity of the body from the axis of rotation. The force of gravity is assumed to produce no rotation. Let the xy-plane be the plane of symmetry and suppose the axes to rotate with the body and the z-axis be the axis of rotation. It is seen that equations (1) and (2) of Art. 104 are now identical and each expresses the fact

$$\Sigma x = \omega^2 M \bar{\rho},$$

and the other four equations are satisfied by this condition.

This will be more easily understood by applying it especially to the sphere in Fig. 125. The only force acting, if ω is constant and the xy-plane is the plane of symmetry, will be a centripetal force $P = \omega^2 M \bar{\rho}$, if we neglect the tendency to rotate about the x and y axes on account of the weight of the sphere.

If the body is also symmetrical with respect to the axis of rotation, we may consider for each half, $P = \omega^2 M \bar{\rho}$,

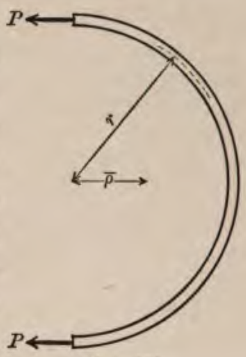


FIG. 131

where M is the mass of $\frac{1}{2}$ of the body and $\bar{\rho}$ is the distance of that one half from the axis of rotation. As an illustration consider the rotation of a fly wheel. Suppose all the centrifugal force is carried by the rim and neglect the spokes. If the mean diameter of the wheel is r (Fig. 131), its mass M, and it rotates with constant angular velocity ω about its axis, then

$$2\,P = \omega^2 M\bar{\rho}.$$

If the rim be considered as a thin wire, $\bar{\rho} = \frac{2\,r}{\pi}$ (Prob. 23) and $M = \frac{\gamma}{g}\,\pi r F$, so that

$$P = \frac{\omega^2 \gamma r^2 F}{g}.$$

It is seen that the tension in the rim varies with the square of the angular velocity.

In particular, suppose the wheel made of cast iron and let $r = 6$ ft. and $F = 10'' \times 4''$. Then

$$P = \omega^2(140.1).$$

If the speed of rotation is six revolutions per second, we have

$$\omega^2 = 1421.29 \text{ and}$$
$$P = 199{,}122 \text{ lb.}$$

Dividing by $F = 40$ sq. in., the area of cross section, we get the stress on the material in pounds per square inch as 4978.

Problem 169. If the wheel just described should "run wild," what speed would be attained before the bursting of the rim occurred, supposing the rim to carry all the centrifugal forces? Assume the tensile strength of cast iron as 25,000 lb. per square inch.

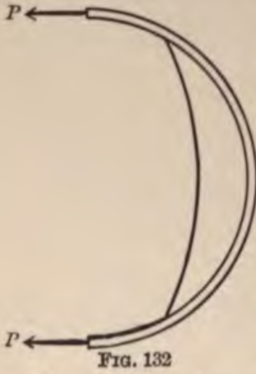


FIG. 132

114. Rotation of a Locomotive Drive Wheel. — The drive wheel of a locomotive, Fig. 132, may be considered for the present as rotating about a fixed axis. We shall consider the effect of the weight of the counterbalance on the tire due to rotation only, on the assumption that the tire carries all the weight of the counterbalance.

NOTE. It is to be understood that the wheel center carries part of the weight of the counterbalance, but a complete solution of the problem of the drive wheel is beyond the scope of this book. The above assumption is therefore made.

Let M be the mass of $\frac{1}{2}$ of tire and $\bar{\rho}$ the distance of its center of gravity from the center of wheel. Let M_1 be the mass of the counterbalance, and $\bar{\rho}_1$ the distance of its center of gravity from the center of wheel.

Then
$$2P = \omega^2 (M\bar{\rho} + M_1\bar{\rho}_1).$$

In particular suppose the diameter of the tread of the tire to be 80 in.; distance of the center of gravity of $\frac{1}{2}$ of tire from center 27 in., and mass of $\frac{1}{2}$ of tire 21. The mass of the counterbalance is 20, and the distance of its center of gravity from the center of the wheel 29 in. Substituting these values, we get

$$2P = \omega^2 \left[21\left(\tfrac{27}{12}\right) + 20\left(\tfrac{29}{12}\right)\right] = 95.5\omega^2.$$

If now we know the speed of rotation of the wheel so that ω is known, we may determine P. Let us take ω corresponding to a speed of train of 60 mi. per hour. This gives $\omega = 26.4$ radians per second and

$$P = 33{,}380 \text{ lb.}$$

Problem 170. If the area of a cross section of tire is 20 sq. in. under the assumption given above, the stress on the metal due to rotation about the axis would be P divided by 20, or 1669 lb. per square inch.

Problem 171. If the allowable stress on the metal is 20,000 lb. per square inch, the value of ω necessary to develop such a stress is given by

$$\omega = \sqrt{\frac{20{,}000 \times 20}{47.7}} = 91.5 \text{ radians per second.}$$

This corresponds to a speed of train of 207 m. per hour.

Problem 172. Consider the rotation to take place about a point on the track. Find P for a speed of train of 60 mi. per hour. Find the corresponding stress in pounds per square inch. What speed of train would develop a stress in the tire of 20,000 lb. per square inch?

115. Rotation about an Axis not a Gravity Axis.—When the center of gravity of a rotating part of a machine is not on the axis of rotation, there is a force tending to bend the shaft equal to $\omega^2 M\bar{\rho}$. The distance $\bar{\rho}$ is the distance of the center of gravity from the axis.

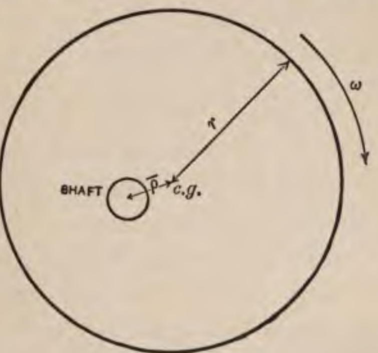


FIG. 133

Suppose the body be a disk of steel, Fig. 133, radius 6 in., and thickness 1 in., and let $\omega = 60\,\pi$ radians per second. If the disk is off center $\frac{1}{2}$ in., the force perpendicular to the shaft due to the unbalanced mass is given by the equation

$$P = \omega^2 M \bar{\rho}$$
$$= (60\,\pi)^2 \pi \left(\tfrac{1}{2}\right)^2 \tfrac{1}{12}\, \tfrac{490}{32.2} \left(\tfrac{1}{24}\right) = 1241 \text{ lb.}$$

Such a disk might be balanced by the addition of a proper weight placed with its center of gravity diametrically opposite the center of gravity of the disk and *in the plane of the disk.*

For static balancing it would not be necessary for the added weight to have its center of gravity in the plane of the disk, but for rotation this is necessary, as will be shown in what follows. Let the shaft AB, Fig. 134, carry weights G and G_1, as shown. When G is up, it tends to lift the end A of the shaft, due to its centrifugal force. When G_1 is up, the end B of the shaft is lifted. Thus for each revolution, the end A is lifted, and then the end B; that is, the shaft *wobbles* about the point C. What really happens is something more than the mere lifting of the shaft in the bearing. Each end describes a cone with C as the apex.

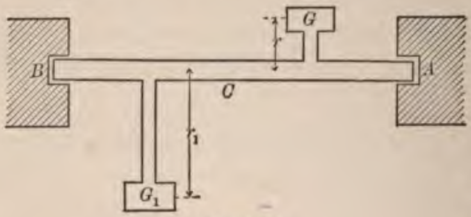


FIG. 134

One wheel, improperly balanced, when rotated on a shaft, causes the shaft to wobble about one of the bearings or to lift bodily. Two wheels improperly balanced on the same shaft, cause the shaft to wobble about some point C.

The principle involved in the balancing of rotating parts is made clear by considering two masses, M_1 and M_2, distant r_1 and r_2, respectively, from the axis of rotation. Suppose these bodies to be in the same plane. For *static balance* it is necessary that $M_1r_1 = M_2r_2$, but for dynamic balance, in addition to this, we must have $M_1r_1^2 = M_2r_2^2$. It follows, therefore, *that for static balance an equivalence of moment is required, while for dynamic balance an equivalence of both masses and distances is required.*

If the counterbalance on the locomotive drive wheel (see Fig. 87) does not balance perfectly the parts on the opposite side of the center and the reciprocating parts, the unbalanced mass will cause the locomotive to lift up when it is above the center. When it is below the center, the weight comes down on the rail with what is known as a "hammer blow." This becomes very destructive to both rail and wheel at high speeds when the unbalanced mass is at all large. It is much the same in effect as the dropping of the weight on the driver through a given distance.

Suppose the counterbalance is too heavy by 64.4 lb., and that its center of gravity is 30 in. from the center of the wheel. If the locomotive is making 60 mi. per hour, and the drivers are 80 in. in diameter, the approximate lifting force when the counterbalance is up is, from $P = \omega^2 M\bar{\rho}$, 3780 lb. For a speed of 100 mi. per hour, this lifting force would be about 10,980 lb. In either case this weight applied suddenly to the rail must be very destructive to both wheel and rail.

Problem 173. A steel disk 3 ft. in diameter and 1 in. thick is not perpendicular to the axis of rotation, but is out of true by $\frac{1}{100}$ of its radius. Find the twisting couple introduced tending to make the shaft wobble.

Problem 174. If the unbalanced weight in a drive wheel in the above illustration is 200 lb., find the centrifugal force for a speed of train of 60 mi. per hour.

116. Rotation of the Fly Wheel of Steam Engine.—Let the fly wheel be given as in Fig. 135 with radius r, and suppose that a belt runs over it horizontally as shown by

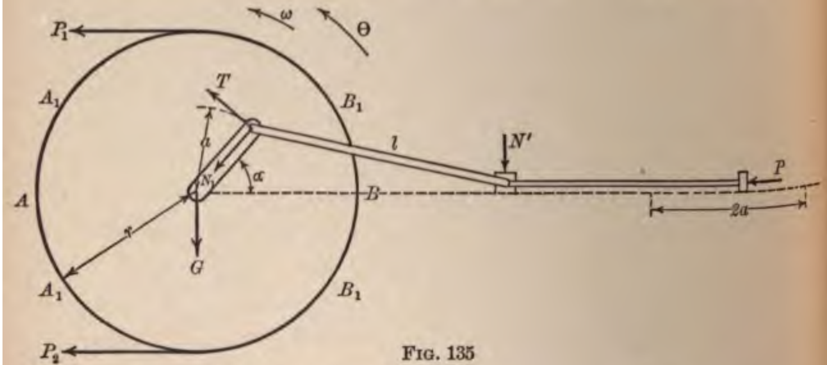


FIG. 135

P_1 and P_2. The effective steam pressure is P; N' is the pressure of the guides on the crosshead. (It is normal if friction is neglected.) The pressure on the crank pin is resolved into tangential and radial components T and N_1. The relation between P_1 and P_2 is shown by the expression $P_1 = (\text{const.})\,P_2 = CP_2$ (see Art. 156). The six general equations give, considering only the fly wheel,

$$\begin{aligned}
\Sigma x &= -P_1 - P_2 - N_1 \cos\alpha - T\sin\alpha = 0,\\
\Sigma y &= T\cos\alpha - N_1 \sin\alpha - G = 0,\\
\Sigma z &= 0,\\
\Sigma\,\mathrm{mom}_x &= 0,\\
\Sigma\,\mathrm{mom}_y &= 0,\\
\Sigma\,\mathrm{mom}_z &= P_1 r - P_2 r + Ta = \theta I_z.
\end{aligned}$$

It is reasonable to assume that the resistance of the machinery, as shown by P_1 and P_2, is constant. The last equation, then, states that T, the tangential force on the crank pin, varies with θ, the angular acceleration. From this equation we have, remembering that $P_1 = CP_2$, $C < 1$ or $P_2 > P_1$,

$$\frac{Ta - (P_2 - P_1)r}{I_z} = \theta.$$

Since $P_2 > P_1$, it is evident that the numerator will be zero when $Ta = (P_2 - P_1)r$, so that θ will be zero for such a case. This makes the angular velocity ω either a maximum or a minimum at such a point. At the dead point B, T is zero; as α increases, T increases until at a certain point B_1 it equals $(P_2 - P_1)\frac{r}{a}$. At this point $\theta = 0$ and ω is a minimum. Beyond B_1, ω increases and θ increases, T has a maximum value and so does θ at some point beyond B_1, after which T decreases, since it is again zero at the dead point A. Thus in passing to zero there is a value such that $T = (P_2 - P_1)\frac{r}{a}$, so that θ is again zero at some point A_1. At this point A_1, θ changes from positive to negative, so that ω is a maximum. It is evident that there are two corresponding points A_1 and B_1 below the line AB.

The above equation may be written

$$\int_{\omega_0}^{\omega} \omega d\omega = \frac{1}{I_z}\int_{\alpha_0}^{\alpha} Ta d\alpha - \frac{(P_2 - P_1)}{I_z}\int_{\alpha_0}^{\alpha} r d\alpha,$$

where the subscript zero indicates some initial value; now $a d\alpha = ds$, distance in the crank-pin circle and $r d\alpha = ds'$, distance in the fly-wheel circle. We may, accordingly, write

$$I_z \frac{\omega^2 - \omega_0^2}{2} = \int_{s_0}^{s} Tds - (P_2 - P_1)\int_{s_0'}^{s'} ds'$$

$$= \int_{s_0}^{s} Tds - (P_2 - P_1) \text{ arc of fly wheel}\Big]_{s_0'}^{s'}.$$

Since work done (see Art. 135) on one end of connecting rod equals the work done on the other, $\int_{s_0}^{s} Tds = \int_{x_0}^{x} Pdx$, where x, x_0, and dx are distances in the cylinder corresponding to s, s_0, and ds in the crank-pin circle. Assuming that this has already been shown, we may write

$$I_z \frac{(\omega^2 - \omega_0^2)}{2} = \int_{x_0}^{x} Pdx - (P_2 - P_1) \text{ arc of fly wheel}\Big]_{s_0'}^{s'}.$$

The approximate value of $\int_{x_0}^{x} Pdx$ may be found by reading from the indicator card the values of P for successive values of x between the limits x and x_0.

A more exact treatment of the above equation may be obtained by considering that the expansion of steam in the cylinder is constant and equal to P' up to the point of cut-off and that beyond this point the pressure varies inversely as the volume. If we assume P constant and equal to P' to the cut-off, then the limits of integration will be regarded accordingly, and we may write

$$I_z\frac{(\omega_1^2-\omega_0^2)}{2}=P'\int_{x_0}^{x_1}dx-(P_2-P_1)\text{ arc of fly wheel}\Big]_{s_0'}^{s_1'}$$

$$=P'(x_1-x_0)-(P_2-P_1)\text{ arc of fly wheel}\Big]_{s_0'}^{s_1'}.$$

Beyond the cut-off P varies inversely as the volume of steam in the cylinder, and so $P=q\left(\frac{1}{\pi r^2x}\right)=\frac{q_1}{x}$. Then from the point of cut-off to the end of the stroke

$$I_z\frac{(\omega^2-\omega_1^2)}{2}=q_1\int_{x_1}^{x}\frac{dx}{x}-(P_2-P_1)\text{ arc of fly wheel}\Big]_{s_1'}^{s'}.$$

If the pressure be regarded as constant throughout, that is, if the mean effective pressure P_1' be substituted for P, we have, considering the motion from B to A,

$$I_z\frac{(\omega_A^2-\omega_B^2)}{2}=P_1'2a-(P_2-P_1)\pi r.$$

The two first equations of this article in ΣX and ΣY give

$$T=G\cos\alpha-(P_2+P_1)\sin\alpha,$$

and

$$N_1=G\left(\frac{\cos^2\alpha-1}{\sin\alpha}\right)-(P_2+P_1)\cos\alpha.$$

Problem 175. Suppose the mean effective steam pressure is 16,000 lb., the radius of the crank-pin circle 18 in., and the radius of the fly wheel 3 ft. If $(P_2-P_1)=500$ lb. and $\omega_B=2\pi$ radians per second, find ω_A; if $I_z=2000$.

Problem 176. The fly wheel in the above problem has a velocity $\omega_A=6\pi$ radians per second. What constant resistance (P_2-P_1) will change this to 2π radians per second in 100 revolutions?

Problem 177. Find the values of α for which ω and θ are maximum and minimum in the above problem.

117. Rotation and Translation. — In this work only the simple case of rotation and translation in a straight line will be considered, since the engineer is not usually concerned with more complicated motions. Let us consider the motion of a body rotating about an axis that moves parallel to itself. Suppose the body in Fig. 136 rotates about O, and at the same time has a motion of translation along X. Take the origin at O, and allow it to be translated with the body. Any elementary mass dM of the body may be considered subjected to a force $dM \cdot a$ parallel to X, due to the translation, a tangential force $dT = \theta M\rho$, and a normal force $dN = \omega^2 M\rho$. These are the effective forces. Writing down the equations of equilibrium between these forces and the impressed forces, we have at any instant

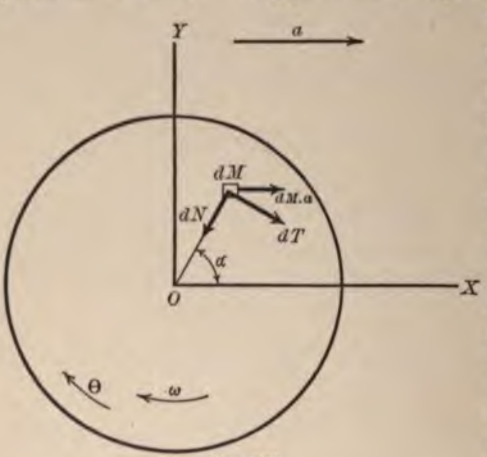


FIG. 136

$$\Sigma x = \int dMa + \int dT \sin\alpha - \int dN \cos\alpha = Ma + \theta M\bar{y} - \omega^2 M\bar{x}$$

$$\Sigma y = -\int dN \sin\alpha - \int dT \cos\alpha = -\omega^2 M\bar{y} - \theta M\bar{x}$$

$$\Sigma z = 0$$

$$\Sigma \text{mom}_x = 0$$

$$\Sigma \text{mom}_y = 0$$

$$\Sigma \text{mom}_z = -\int dT\rho - \int dMa\rho \sin\alpha = -\theta I_z - aM\bar{y}.$$

It is seen at once that if the x and y axes pass through the center of gravity, so that $\bar{x}$ and $\bar{y}$ are each zero, the right-hand side of the second equation becomes zero, and the first and last equations may be written

$$\Sigma x = Ma,$$
$$\Sigma \operatorname{mom}_z = -\theta I_z.$$

As an illustration let us assume that a cast-iron cylinder rolls on a straight horizontal track, due to the application of certain impressed forces. Suppose the radius of the cylinder is 18 in. and its thickness 2 in., and that at the time of observation it is making 10 revolutions per second. From this time there is only a constant tangen-

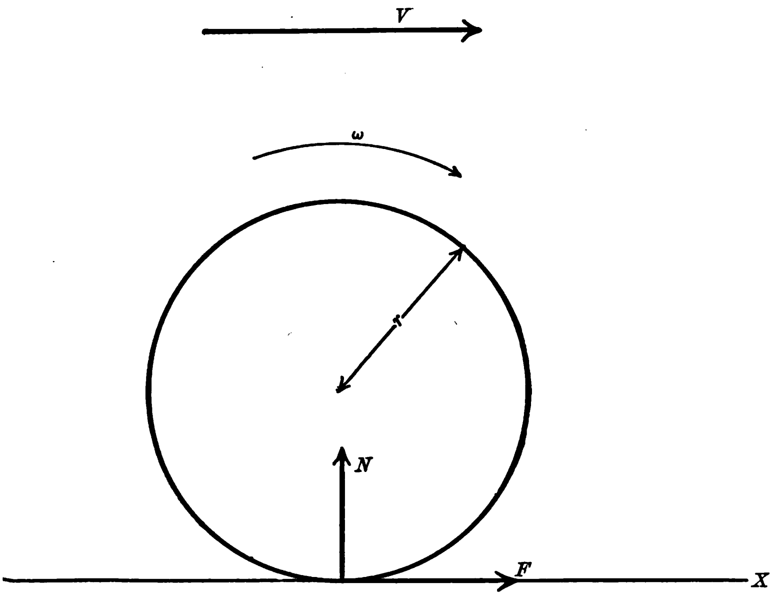


FIG. 137

tial force of friction F acting parallel to X and the normal pressure N. The cylinder comes to rest in one minute. Find F; the distance passed over, in coming to rest; the angular velocity at the end of 10 sec., and the linear velocity at the end of 30 sec. Take the origin at the center and X horizontal. From Fig. 137 it is seen that the general equations for this case become

$$\begin{aligned} F &= Ma, \\ N &= G, \\ Fr &= \theta I_z. \end{aligned}$$

Since F is constant, a and θ are constant, so we have in addition to the above equations

$$\begin{aligned} \omega &= \omega_0 + \theta t, \\ \alpha &= \frac{\omega^2 - \omega_0^2}{2\,\theta}, \\ \alpha &= \tfrac{1}{2}\,\theta t^2 + \omega_0 t. \end{aligned}$$

These equations are sufficient to determine the unknown quantities.

Problem 178. The same cylinder given in the above illustration rolls, without slipping, down a rough inclined plane, inclined at an angle δ to the horizontal. In addition to the forces acting as given above there is a component of gravity (see Fig. 138).

Let x be parallel to the plane, then

$$\begin{aligned} \Sigma x &= F + G \sin \delta = Ma, \\ \Sigma y &= N - G \cos \delta = 0, \\ \Sigma \operatorname{mom}_z &= Fr = \theta I_z, \end{aligned}$$

so that a is constant and equal to $\left[\dfrac{g \sin \delta}{1 + \dfrac{k_z^2}{r^2}}\right]$.

If the cylinder has the same velocity as in the above illustration, find the constant force of friction and the distance passed over in coming to rest, $\delta = 10°$.

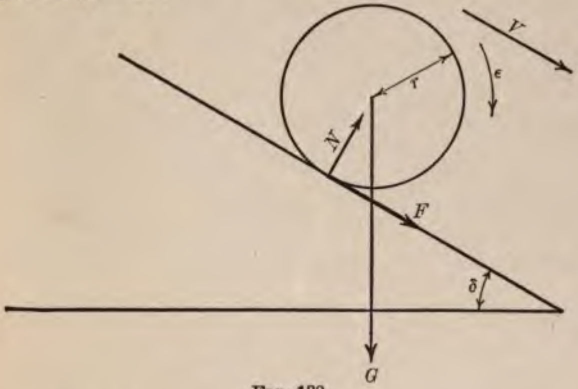


Fig. 138

Problem 179. A cast-iron cylinder, radius 3 in. and 6 in. long, rolls down a rough inclined plane, inclined at an angle of 30° with the horizontal. Find the acceleration down the plane; the angular acceleration; the force of friction F; and the normal pressure N.

118. Side Rod of Locomotive. — The side rod of a locomotive furnishes an interesting study of a case of combined rotation and translation. Assume the velocity of the locomotive uniform so that each dM of the side rod revolves uniformly in the arc of a circle of radius r. See Fig. 139 (referred to the locomotive). Writing the equation of equilibrium after

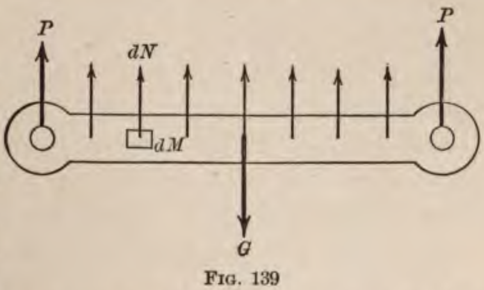


Fig. 139

neglecting the thrust due to the pressure of steam on the piston, we have

$$2P - G = \int dN = \int \frac{v^2}{r} dM = \frac{Mv^2}{r},$$

where P is the pressure on a crank pin due to the rotation alone and v is the tangential velocity of any dM relative to the locomotive. If v' is the velocity of the train and r' the radius of the drive wheel, then

$$v' = \frac{r'}{r} v,$$

so that

$$2P - G = \frac{Mrv'^2}{r'^2}.$$

Problem 180. Suppose the locomotive to have a velocity of 90 mi. per hour, the radius of the crank-pin circle 20 in., the radius of the drive wheel 40 in., and the weight of the parallel rod 400 lb. Find the pressure on the crank pins due to the centrifugal force.

119. The Connecting Rod. — The connecting rod of an engine has a circular motion at one end while the other end moves backward and forward in a straight line. We shall consider the motion relative to the engine and shall assume that the fly wheel is of sufficient weight to give the crank a motion sensibly uniform. It will be convenient to regard the motion of the connecting rod as consisting of a rotation about the crosshead end while that end is moving in a straight line.

In Fig. 140 let A be the crosshead and O the center of the crank-pin circle. Let l be the length of the connecting rod, and r the radius of the crank-pin circle. If we neglect friction, the only forces acting on the connecting rod at A are N', the pressure of the guides, and P', the pressure exerted by the piston rod. The force exerted on the connecting rod by the crank pin has been resolved into its normal and tangential components N_1 and T, respectively. Suppose ω_1 to be the constant angular velocity of the crank, and suppose the angular velocity of the rod about A to be represented by ω and the angular acceleration by θ.

If we consider any element of mass dM of the rod, it is seen that the forces acting upon it consist of a force $dP =$ $dM \cdot a$ parallel to OX, a normal force $dN = \omega^2 M\rho$ and a tangential force $dT = \theta M\rho$, so that at any instant we have the same formulæ as those developed in Art. 104. These may be written in this case.

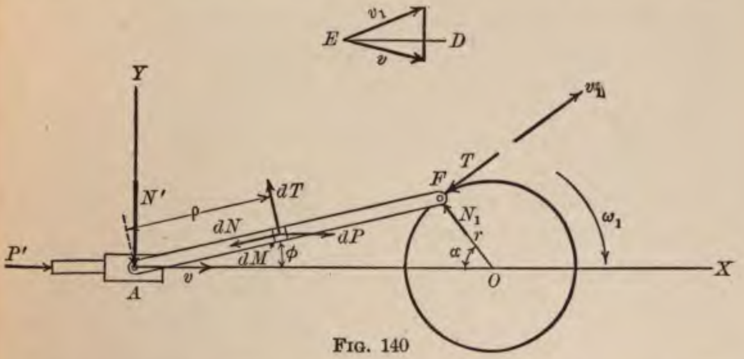


FIG. 140

$$\Sigma x_i = P' - T\sin\alpha - N_1\cos\alpha = Ma - \omega^2 M\bar{x} - \theta My,$$
$$\Sigma y_i = -N' + N_1\sin\alpha - T\cos\alpha = -\omega^2 M\bar{y} + \theta M\bar{x},$$
$$\Sigma\,\text{mom}_{zi} = N_1 l\sin(\alpha+\phi) - Tl\cos(\alpha+\phi) = \theta I_z - aM\bar{y}.$$

The axis of rotation cannot pass through the center of gravity, so no further reduction is possible.

We know

$$\frac{\sin \alpha}{\sin \phi} = \frac{l}{r},$$

so that
$$\phi = \sin^{-1}\left(\frac{r}{l} \sin \alpha\right);$$

differentiating with respect to t,

$$\frac{d\phi}{dt} = \omega = \frac{\frac{d}{dt}\left(\frac{r}{l} \sin \alpha\right)}{\sqrt{1 - \frac{r^2}{l^2} \sin^2 \alpha}};$$

therefore
$$\omega = - \frac{r\omega_1 \cos \alpha}{\sqrt{l^2 - r^2 \sin^2 \alpha}},$$

since
$$\frac{d\alpha}{dt} = - \omega_1$$

and
$$\theta = \frac{- \omega_1^2 r \sin \alpha (l^2 - r^2)}{(l^2 - r^2 \sin^2 \alpha)^{\frac{3}{2}}}.$$

From these last two equations, for any values of ω_1 and α we may obtain ω and θ. The linear acceleration, a, has been taken the same for each dM of the rod, and equal to the acceleration of the piston. If the quantities M, $\bar{x}$, $\bar{y}$, and I_z are known; we might find for any steam pressure P' and the corresponding a, the forces N', N_1, and T. In other words, we could determine the forces acting on the guides and crank pin.

Problem 181. The connecting rod given in Problem 163, Art. 108, is in use on an engine whose crank has a constant angular velocity of 26 radians per second. The length of the crank is 2 ft., the effective steam pressure on the piston is 16,000 lb. Let α be taken as 30°, then $\phi = 5° 45'$; $\omega = - 4.52$ radians per second, and $\theta = - 69$ radians per second squared. From Problem 163, $I = 2172$, $M = 61.1$, $\bar{x} = 5.3$ ft., $\bar{y} = .533$ ft. To determine a, consider the relation between the velocity v of the piston and the velocity v_1 of the crank pin, Fig. 140. Since the velocity of every point of the rod in the direction of its length is the same, the projections of v and v_1 on the rod are equal. The relations will not be altered if the figure vv_1ED be turned through 90° and E be placed at O and v_1 be made to coincide with OF and drawn to such a scale as to equal r. Then v will fall on OG and will be of a length equal to that cut off on OG by AF produced (see Fig. 141).

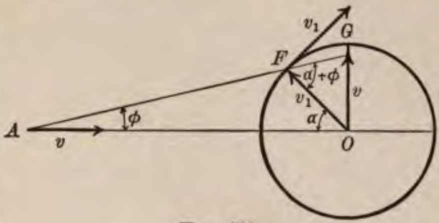


FIG. 141

Then $$\frac{v}{v_1} = \frac{\sin(\alpha+\phi)}{\cos\phi} = \sin\alpha + \cos\alpha\tan\phi,$$
$$v = v_1(\sin\alpha + \cos\alpha\tan\phi).$$

But $\frac{\sin\phi}{\sin\alpha} = \frac{r}{l}$. For small values of ϕ we may replace $\tan\phi$ by $\sin\phi$, so that

$$v = v_1(\sin\alpha + \cos\alpha\frac{r}{l}\sin\alpha) = v_1(\sin\alpha + \frac{r}{2l}\sin 2\alpha).$$

The acceleration $a = \frac{dv}{dt}$

$$= v_1(\cos\alpha + \frac{r}{l}\cos 2\alpha)\omega_1.$$

But since $$v_1 = \omega_1 r,$$
$$\boldsymbol{a} = \boldsymbol{\omega_1^2 r}(\mathbf{cos}\,\alpha + \frac{\boldsymbol{r}}{\boldsymbol{l}}\mathbf{cos\,2}\,\alpha).$$

For $a = 30°$, $v = -30.5$ ft. per second, and $a = 1306$ ft. per (second)^2. The three equations in Σx, Σy, and $\Sigma\,\text{mom}_z$ now give $N' = 9660$ lb., $N_1 = 58{,}137$ lb., and $T = -18{,}151$ lb. Compounding N_1 and T, we get the resultant pressure on the crank pin to be 60,900 lb. The negative signs for N_1 and T indicate that the arrows were assumed in the wrong direction.

Problem 182. Show that the values of α that make ω a maximum or minimum, when the motion of the crank is assumed constant, are π, 3π, etc., and o, 2π, etc.

Problem 183. Find what values of α will make θ a maximum or minimum. Locate the crosshead for these values.

Problem 184. Find values for T, N', N_1, and the resultant pressure on the crank-pin when $\alpha = \pi$ and when $\alpha = 0$. Use the above data.

Problem 185. Assume a force of friction F acting on the crosshead, such that $F = .06\ N'$. In the above case when $\alpha = 30°$, what is the value of F, N', N_1, and T?

Problem 186. Suppose the steam pressure zero, find T, N', N_1, and the resultant crank-pin pressure, if ω_1 is the same.

120. Body Rotating about an Axis — One Point Fixed. — We shall now consider the case of a body rotating about an axis when only one point of that axis is fixed. Consider the equations 1, 2, 3, 4, 5, and 6 (Art. 104) and recall that a body acted upon by any system of forces may be considered as being acted upon by a single force and a single couple (Art. 36). If one point of the axis is fixed, the single force will act at this point, but the effect of the single couple will be to move the axis of rotation about this point. The components of the moment of the couple are shown in the right-hand side of equations (4) and (5). If these reduce to zero, the couple vanishes and rotation continues about the original axis of rotation even though only one point of that axis is fixed. But this can happen only when $\int xzdM = 0$ and also $\int yzdM = 0$. We may say, then, *that a body rotating about an axis one point of which is fixed, when no forces are acting to produce rotation, will continue to rotate about that axis with a constant angular velocity* ω.

121. Gyroscope. — The gyroscope illustrated in Fig. 142 consists of a metal wheel A mounted on an axis BB, fixed at one point to the stand *C*. The weight *D* serves to balance the wheel about the point of support. The wheel *A* is very delicately mounted, so that there is little friction. It is set in motion by means of a cord wound around its axle, as in the case of an ordinary top.

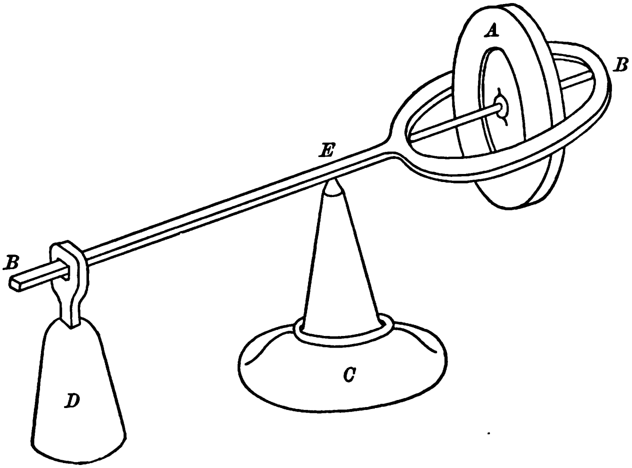


FIG. 142

When the weight *D* exactly balances *A* so that *BB* is horizontal, we have a case of a body rotating about an axis fixed at one point, with no external forces acting. According to the previous article, the body continues to rotate about that axis.

If, however, the weight *D* does not balance *A*, so that *BB* is not horizontal, the axis of rotation changes, since in that case the force of gravity tends to turn *BB* about a horizontal axis through *E* perpendicular to *BB*. As a result of the two rotations, the body tends to turn about a new axis, so that *BB* turns about the point *E* horizontally. All this is in accordance with Art. 95, where we saw that the resultant of two angular velocities was an angular velocity given by the diagonal of a parallelogram constructed upon the two velocity arrows as sides. This rotation of the gyroscope about the vertical axis through E is known as *precession*. The student may reproduce the above results experimentally by taking a bicycle wheel mounted upon its axle. Suspend one end of the axle by a string and hold the other in the hand, so that the axle is horizontal. With the other hand now give the wheel a spin. If the axle remains horizontal, the wheel continues to spin about the same axis, but if the hand supporting one end of the axle be removed, the wheel continues to rotate about its own axis while the axis rotates about the suspending string. In other words, the wheel has a motion of precession.

The motion of the bicycle wheel is explained in the same manner as that used in explaining the precession of the gyroscope.

122. The Spinning Top. — The student will be interested in seeing that the motion of a spinning top, with which all are familiar, is also capable of the same explanation. Let the top be represented, as in Fig. 143, with its point at O, and suppose it has an angular velocity ω about its own axis. If it is rotating sufficiently rapidly, and its axis is vertical, it "sleeps," or continues to revolve about that same axis. If, however, the axis be tilted slightly from the vertical, the weight of the top G and the reaction of the floor constitute an unbalanced couple tending to make it revolve about an axis through O perpendicular to the paper. The result of these two rotations is to cause the top to always tend to revolve about an axis a little in front of the geometrical axis. This causes the axis of spin to continually advance, and describe a cone about OZ, that is, the top precesses. It is well known that when the top has been so disturbed, if spinning rapidly, it moves about OZ very slowly, and gradually takes the vertical position again.

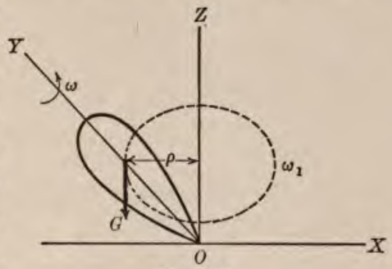


FIG. 143

When the velocity of rotation ω finally becomes small, the irregularities of the support throw the axis out of the vertical, and the action of the unbalanced couple causes precession. At first the precession is very slow, but gradually increases as ω decreases until the top falls.

123. Motion of Earth. — A brief presentation of this subject would be incomplete without mentioning the precession of the earth's axis. In Fig. 144 let S represent the sun and E the earth, with its axis slightly inclined to the vertical. The earth is a spheroid with its axis of rotation as its short axis. Consider the ring of matter near the equator which if cut off would make the earth spherical. The attraction of the sun for this ring of matter is greater on the side nearest the sun. This causes the earth to be acted upon by an unbalanced force F, and tends to cause a rotation of the earth about an axis through O perpendicular to the paper. As a result the axis of rotation is moved forward slightly so that its path is a cone about OZ. On account of the very great velocity of the earth and the smallness of the force F, this precession is very slow, a complete rotation of the axis about OZ requiring 25,800 years. (See Young's "General Astronomy," Precession of the Equinoxes.)

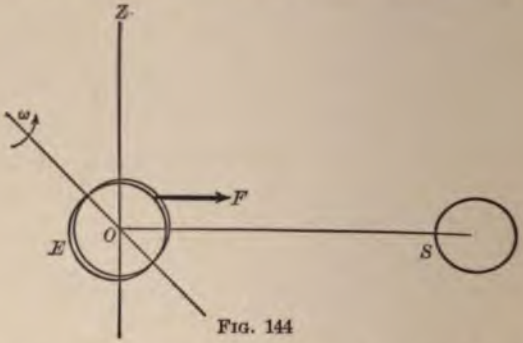


FIG. 144

124. Plane of Rotation. — We have seen that a body moving in a straight line continues to move in that line due to its inertia unless acted upon by some force (Art. 76). In a similar way a body rotating about an axis tends to maintain its axis and plane of rotation due to its *moment of inertia* unless acted upon by some external forces. The moment of inertia of a rotating body has the same relation to its rotation as the inertia of a body has to its translation.

The student may get some idea of this tendency of a rotating body to maintain its plane and axis of rotation from the study of a bicycle wheel mounted on its axle. If the wheel be held by grasping both ends of the axle and if it is then rotated rapidly, it will be found that there is no difficulty in moving the rotating wheel in the plane of rotation, but that as soon as an attempt is made to change the direction of the axle, there is considerable resistance. The same experience may be had by treating the wheel of the gyroscope (Art. 121) in a similar way.

The same action takes place in the rolling of ships at sea. The rolling is lessened by the tendency of the large fly wheels on board to maintain their plane of rotation.

125. Gyroscopic Action Explained. — Spinning bodies, such as the gyroscope or a fly wheel, when acted upon by an unbalanced couple that produces a rotation at right angles to the spin, rotate about an axis at right angles to each of these (Art. 124). Such action may be explained as follows (see Fig. 145). Suppose the body to be a disk,

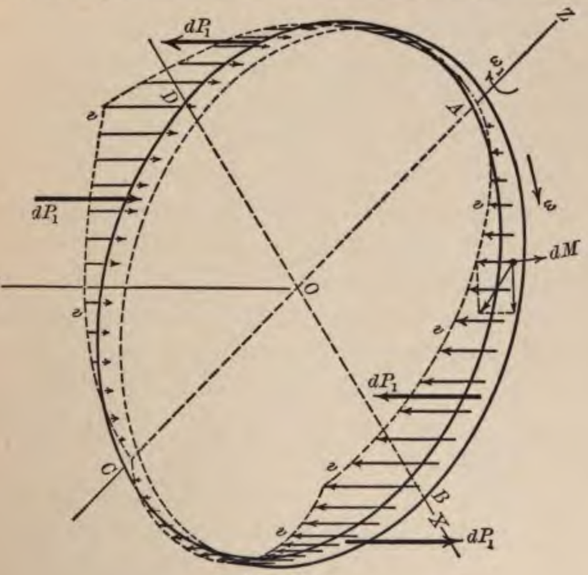


FIG. 145

and that rotation is taking place about the axis OY in the direction indicated. A couple shown by the forces P produces rotation about the axis OZ in the direction indicated.

Consider any element of the disk dM; it is subjected to two angular velocities, ω about OY, and ω_1 about OZ. If we imagine the element at the point A on the axis OZ, its velocity v due to rotation about OZ is $\rho\omega_1 = 0$, since $\rho = 0$. As it moves from A toward B this velocity v increases, and from B to C it decreases until it is zero at C. This increase and decrease of the velocity is represented in the figure by the arrows v. Since the velocity of dM increases in going from A to B, the increase must be caused by some force having the direction dP_1, on the side facing the reader. The decrease in the velocity v in going from B to C must be due to a force dP_1, acting away from the side facing the reader. In a similar way the velocity v increases in going from C to D and decreases in going from D to A. The force acting on any dM as it moves from A to A again are represented by the arrows dP_1. It is seen that the result of such forces acting on every dM will be to turn the disk about the axis OX. This motion about OZ we have called precession (Art. 121 and Art. 122).

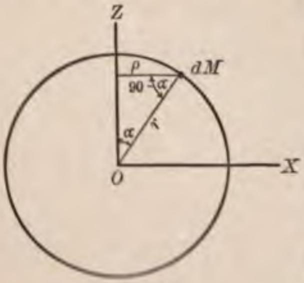


Fig. 146

126. Precessional Moment; Special Case.

— A simple analysis serves to give the moment about the axis OX. Let the disk of the preceding article be represented in Fig. 146 with the axis OY

perpendicular to the paper. The elementary mass dM is in the position shown. The velocity (v) of this dM is

$$v = \rho\omega_1 = r\omega_1 \sin\alpha,$$

when ρ is the distance from the axis OZ, ω_1 the angular velocity about OZ, and r is the distance from O.

Therefore the acceleration a_1 in the direction of v is given by

$$\frac{dv}{dt} = r\omega_1 \cos\alpha \frac{d\alpha}{dt}.$$

But $\frac{d\alpha}{dt} = \omega$, the angular velocity about OY,

so that $$a_1 = r\omega_1\omega \cos\alpha$$

and dP_1, the force in the direction of v,

$$= dM \cdot a_1 = dMr\omega_1\omega \cos\alpha.$$

We may write $$dM = t\frac{\gamma}{g} r d\alpha\, d\rho.$$

Then $$dP_1 = t\frac{\gamma}{g}\omega_1\omega r^2 dr \cos\alpha d\alpha.$$

The moment of this force about the axis OX is

$$\text{mom}_{ox} = dP_1(r\cos\alpha) = t\frac{\gamma}{g}\omega_1\omega r^3 dr \cos^2\alpha d\alpha.$$

The moment for the whole disk U about the axis OX is then

$$U = t\frac{\gamma}{g}\omega_1\omega\int_0^r r^3 dr\int_0^{2\pi}\cos^2\alpha d\alpha$$

$$= t\frac{\gamma}{g}\omega_1\omega\frac{r^4}{4}\int_0^{2\pi}\cos^2\alpha d\alpha = t\frac{\gamma}{g}\omega_1\omega\frac{r^4}{4}\left[\frac{\alpha}{2} + \frac{1}{4}\sin 2\alpha\right]_0^{2\pi}$$

$$= t\frac{\gamma}{g}\omega_1\omega\frac{r^4}{4}\pi.$$

But
$$M = t\frac{\gamma}{g}\pi r^2,$$

so that
$$U = M\omega_1\omega\frac{r^2}{4}.$$

If the rotating body is the rim of a fly wheel with outside radius r_1 and inside r_2, the value of U changes only in integrating between the limits r_1 and r_2. Then

$$U = t\frac{\gamma}{g}\omega_1\omega\frac{(r_1^4 - r_2^4)}{4}\pi = M\omega_1\omega\frac{(r_1^2 + r_2^2)}{4}.$$

As an illustration, suppose the weight of the ring of a gyroscope to be 50 lb. and the mean radius 6 in., and outside radius 8 in., and let ω_1 be unity, and suppose it makes 100 revolutions per second about the axis OY. Then

$$U = \frac{50}{32.2} \times 200\,\pi \times \frac{5}{36} = 146 \text{ ft. lb.}$$

It is seen that with a small value for ω_1 the tendency to turn about OX is considerable.

NOTE. The above analysis is substantially the same as that given in *Engineering*, June 7, 1907.

Problem 187. A ship carries a cast-iron fly wheel whose rim is 6 ft. outside diameter, 4 in. thick, and 18 in. wide. When it is making 3 revolutions per second, its axis is turned about an axis through the plane of the wheel with unit angular velocity. Find the moment of the couple that tends to turn the wheel about an axis perpendicular to these two axes.

Problem 188. A solid cast-iron disk 3 ft. in diameter and 3 in. thick revolves about its axis, making 3000 revolutions per minute. At the same time it is made to turn about an axis in its plane at the rate of 2 revolutions per minute. Find the magnitude of the couple tending to rotate the disk about an axis perpendicular to these two axes.

The equation $$U = M\omega_1\omega\frac{r^2}{4}$$

may be written $$\frac{U}{\omega} = \frac{Mr^2}{4}\omega_1.$$

If the value of the couple U is constant, it is seen that ω varies inversely with ω_1. That is, if ω is large ω_1 is small, and if ω is small ω_1 is large. This was pointed out in Art. 122 and illustrated in the case of the top when it is dying down. As the spin decreases in such cases, the precession increases.

127. Precessional Moment; General Case. — The precessional moment for any body when OZ is perpendicular to OY may be obtained from a consideration of the moment equation (Art. 126) by retaining the dM. The equation may then be written

$$U = \int dP_1(r \cos \alpha) = \int dM\omega_1\omega r^2 \cos^2 \alpha$$

or $$U = \omega_1\omega\int dM(r \cos \alpha)^2.$$

Since $r \cos \alpha$ is the distance of dM from OX, this may be written

$$U = \omega_1\omega I_{ox}.$$

When the axis of precession OZ is inclined at an angle δ to the axis of spin OY, the above value for U must be

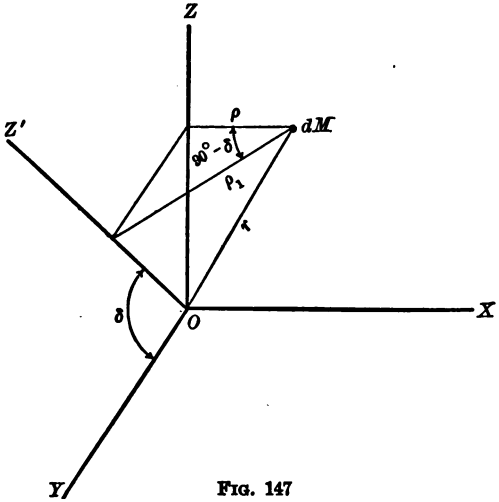


FIG. 147

changed slightly. To make the ideas clear, let the original axis OZ be drawn as well as the inclined axis OZ' making the angle δ with OY, as in Fig. 147. Let ρ be the distance of an elementary dM from OZ, as before, and ρ_1 the distance of dM from OZ'. Then $\rho = \rho_1 \sin \delta$, and the expression for the velocity, $v = \rho\omega_1 = r\omega_1 \sin \alpha$ becomes

$$v = r\omega_1 \sin \alpha \sin \delta.$$

and

$$a_1 = \frac{dv}{dt} = r\omega_1\omega \sin \delta \cos \alpha,$$

since $\sin \delta$ is constant, so that

$$dP_1 = a_1 dM = dMr\omega_1\omega \sin \delta \cos \alpha;$$

therefore

$$U = \int dP_1(r \cos \alpha) = \int dM\omega_1\omega \sin \delta r^2 \cos^2 \alpha$$

$$= \omega_1\omega \sin \delta \int dM(r \cos \alpha)^2,$$

or

$$U = \omega_1\omega I_{ox} \sin \delta.$$

The gyroscopic action of the fly wheel of an automobile has much to do in causing the machine to overturn when rounding sharp curves at high speeds. Even when the machine is not overturned, the gyroscopic moment due to the rotation of the wheel causes an extra pressure on the bearings. This pressure is shown by the wear on the bearings. It is left as an exercise to determine the overturning moment due to the action of an automobile fly wheel under assumed conditions. It will also interest the student to know that a German torpedo boat 116 ft. long, and 56 tons displacement, was held upright in a heavy sea by an 1100 lb. disk rotating 1600 revolutions per minute.

Problem 189. A locomotive is going at the rate of 40 mi. per hour around a curve of 600 ft. radius. The diameter of the drivers is 80 in., and a pair of drivers and axle have a moment of inertia about an axis midway between the wheels and perpendicular to the axle of 3000. What is the magnitude of the couple introduced by the precessional motion of this pair of wheels? Give the direction in which it acts. Does it tend to make the locomotive tip inward or outward?

Problem 190. A car pulled by the locomotive in the preceding problem has four pairs of wheels. The moment of inertia of each pair of wheels and their connecting axle, with respect to an axis midway between the wheels and perpendicular to the axle, is 320 (see problem 87). What is the magnitude of the precessional couple acting upon the whole car?

Problem 191. The fly wheel of an engine on board a ship makes 300 revolutions per minute. The rim has the following dimensions: outside radius 4 ft., inside radius $3\frac{1}{2}$ ft., width 12 in. The ship rolls with an angular velocity of $\frac{1}{2}$ a radian per second; find the torque acting on the ship due to the gyrostatic action of the fly wheel.

Problem 192. A conical top is made of wood and is spinning about its axis with a velocity of 20 revolutions per second. The cone has a base of 2 in. and a height of 2 in., and spins on the apex. While spinning steadily with its axis vertical (sleeping), it is disturbed by a blow so that its axis is inclined at an angle of 30° with the vertical. Find the velocity of precession and the torque U that tends to keep the top from falling. See Fig. 143.

128. Car on Single Rail. — An interesting application of the gyroscope has been made recently in England. A car (see Fig. 148) is run upon a single rail, and is held upright by means of rapidly rotating fly wheels. Each car contains two of these wheels rotating in opposite directions, at the rate of 8000 revolutions per minute.

Any tendency of the car to tip over, either when running or standing at the station, is righted by the gyroscopic action of the fly wheels. The experimental car was so successful in operation that it maintained itself in an upright position even when loaded eccentrically. The action of the fly wheels is such as to place the center of gravity of the car and load directly over the rails.

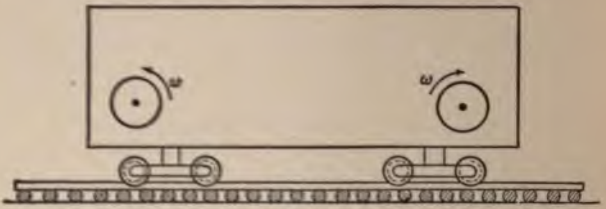


FIG. 148

(NOTE. See *Engineering*, June 7, 1907.)

CHAPTER XIII

WORK AND ENERGY

129. Definitions. — When the forces acting upon a body cause a motion of that body, work is done. We define the work done by a force as *the magnitude of the force times the distance through which the body, upon which it acts, moves along its line of action.*

This definition may be less exactly stated by saying that a force acting on a body that moves through a distance does work. This brings to mind the forces considered as acting in Chapters II, III, and VI, where no motion was produced; that is, where the point of application did not move. Such forces produce no work according to our definition. To make the idea of work clearer, suppose the body C (Fig. 149) to be acted upon by a force P, and that the body is moved until the point at A is finally at B. The work done by P is P *times* AB. Suppose the plane upon which C moves is rough, so that it offers a resistance F. In passing over the distance AB, the force F does a *work of resistance* equal to F *times* AB.

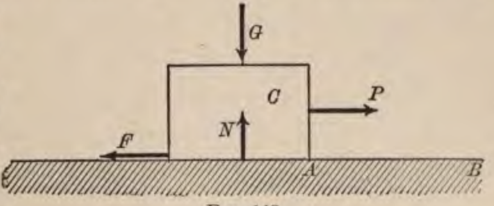


FIG. 149

The upward force N, which is the pressure of the supporting surface, does no work, since no motion takes place along its line of action. The idea of work is related to that of the motion of the body in the direction of the acting force, but is independent of time.

130. Units of Work. — Since work involves force times distance, we express it in terms of the units of force and distance; that is, in *inch-pounds* or *foot-pounds*. These are the units used by engineers in this country and England, and are the units that will be used in this book.

We might say, then, that the *unit of work, the foot-pound, is the work done by a force whose magnitude is one pound when the body upon which it acts moves through a distance of one foot.*

In countries where the metric system is used, the erg is used when a small unit of work is convenient. The *erg* is the work done by a force of one dyne when the body upon which the force acts moves a distance of one centimeter in the direction of the force. A larger unit of work, the *joule*, is often used; the joule is 10^7 ergs. Engineers often use the *kilogram-meter* as a unit of work. It is the work done by a force whose magnitude is one kilogram, while the body upon which the force acts moves one meter in the direction of the force.

131. Graphical Representation of Work. — Work has been defined as the product of a force and a distance. If the force be *uniform* and equal to P, and the body upon which it acts be moved through a distance a, the graphical representation of the work done by P is given by the area of a rectangle, Fig. 150, constructed on P and a as sides, since

$$W = Pa.$$

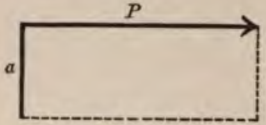


FIG. 150

If the force P varies as the distance through which the body is moved along its line of action, we may represent the work by the area of the triangle as shown in Fig. 151. Let the force be zero when the motion begins, and let it be P_1 when the distance passed over along its line of action is OA. Then since the force varies as the distance, it is equal to P for any intermediate distance OC. The total work done, then, in moving the body through a distance OA by the variable force P, which varies as the distance, is equal to $\frac{P_1(OA)}{2}$. It is seen that this is the same as the work done by the average force $\frac{P_1}{2}$ acting through the distance OA. The resistance of a helical spring varies with the elongation or compression. The same law of variation holds for all elastic bodies.

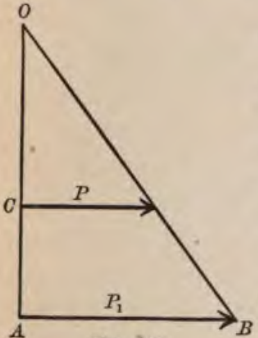


FIG. 151

Another variation of force with distance with which the engineer is frequently concerned, is the case where the force *varies inversely as the distance* through which the body is moved. If P is the force and S the distance, the relation between force and distance may be expressed, $P = \frac{\text{const.}}{S}$, or $PS = \text{const.}$ But this represents the equilateral hyperbola. This will be made clearer by reference to the specific example of the expansion of steam in a steam cylinder. (See Fig. 152.) Up to the point of cut-off *C*, the steam pressure is the same as that in the boiler (practically), and is constant while the piston moves from *O* to *C*. At this point, the entering steam is cut off and the work done must be done by the expansion of the steam now in the cylinder. According to Mariotte's Law, the pressure varies inversely with the volume of steam; but since the cross section of the cylinder is constant, we may say that the pressure varies inversely as the distance. From the properties of the curve, it is easy to see that the area under the curve represents the work done.

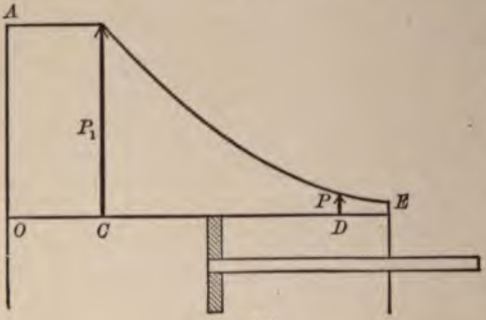


FIG. 152

The curve obtained in practice representing the relation between the force and distance is shown in Fig. 153.

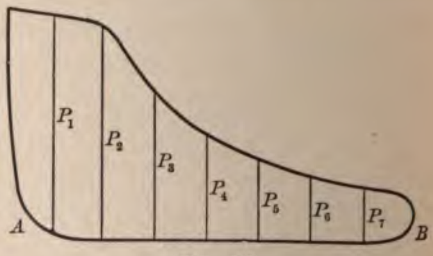


FIG. 153

The curve after cut-off is not a true hyperbola, and its area is determined by means of a planimeter or by Simpson's Rule.

132. Power.— The idea of work is independent of time. But for economical reasons it is necessary to take into consideration this element of time. We must know whether certain work has been done in an hour or ten hours. For such information a unit of the rate at which work is done has been adopted. This unit is called power. *Power is the rate of doing work. It is the ratio of the work done to the time spent in doing that work.*

The unit of power is the *horse power*. This has been taken as 550 ft.-lb. per second, or 33,000 ft.-lb. per minute. Originally the idea of the rate of work was connected with the rate at which a good draft horse could do work. This value as used by Watt was 550 ft.-lb. per second. The horse power of a steam engine is mean effective pressure times distance traveled by the piston per second, divided by 550.

133. Energy. — *Energy is the capacity for doing work;* it is stored-up work. Bodies that are capable of doing work due to their position are said to possess *potential energy*. Bodies that are capable of doing work due to their motion are said to possess *kinetic energy*. A familiar example of potential energy is the energy possessed by a brick as it is in position on the top of a chimney. If the brick should fall, its energy at any instant would be called kinetic. When the brick strikes the ground, work is done in deforming the ground and brick, or perhaps even breaking the brick and even generating heat. The work done by the brick when it strikes is sufficient to use up all the energy that the brick had when it struck.

134. Conservation of Energy. — The kinetic energy of the brick spoken of in the last article was used up in doing work on the ground and air, and upon the brick itself, so that the kinetic energy that the brick possessed when it struck was used up. It was not, however, destroyed, but was transferred to other bodies, or into heat. Such transference is in accord with the well-known principle of the conservation of energy. This principle may be stated as follows: *energy cannot be created or destroyed.* The amount of energy in the universe is constant. This means that the energy given up by one body or system of bodies is transferred to some other body or bodies. It may be that the energy changes its form into light, heat, or electrical energy.

Energy cannot be created or destroyed; it is, therefore, evident that such a thing as *perpetual motion* is impossible. Such a motion would involve the getting of just a little more energy from a system of bodies than was put into them.

135. Energy of a Body moving in a Straight Line. — Suppose the body, Fig. 154, moving in a straight line as indicated with a variable velocity v. Let P be a constant working force, so that the resulting force acting on the body will be $P-R$. From the relation that the accelerat-

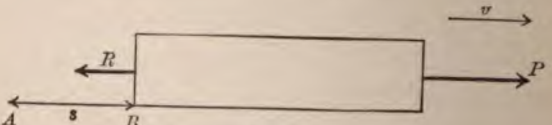


Fig. 154

ing force equals the mass times the acceleration, we may write

$$vdv = \frac{(P-R)}{M}ds.$$

Integrating between the limits v_0 and v, and 0 and s, we have

$$\frac{Mv^2}{2} - \frac{Mv_0^2}{2} = (P-R)\,s,$$

or

$$\boldsymbol{\frac{Mv^2}{2} - \frac{Mv_0^2}{2} + Rs = Ps.}$$

The quantity $\frac{Mv^2}{2}$ is the kinetic energy of the body when it has a velocity v, and the quantity $\frac{Mv_0^2}{2}$ the kinetic energy of the body when it has a velocity v_0. The left-hand side of the equation, therefore, represents the change in kinetic energy. The equation shows that *the work done by the working force equals the work done by the resisting force plus the change in the kinetic energy.*

The weight of the body is 64.4 lb., and P is a constant force, say 100 lb., and R a constant resistance = 84 lb. If the body starts from rest, what will be the velocity when it has moved a distance of 16 ft.?

Substituting in the above equation, we have $v = 16$ ft. per second.

Problem 193. A car whose weight is 20 tons, and having a velocity of 60 mi. per hour, is brought to rest by means of brake friction after the power has been shut off. If the tangential force of friction of 200 lb. acts on each of the 8 wheels, how far will the car go before coming to rest?

SOLUTION. Here $M = \frac{40{,}000}{32.2}$, $v_0 = 88$ ft. per second, $v = 0$, $P = 0$, since there are no working forces. $R = 1600$ lb., so that

$$\frac{40{,}000}{2(32.2)}(88)^2 = 1600\,s.$$

Therefore $$s = 3009 \text{ ft.}$$

Problem 194. Suppose the car in the preceding problem to be moving at the rate of 60 mi. per hour when the power is shut off, what tangential force on each of the 8 wheels will bring the car to rest in one half a mile?

Problem 195. What is the kinetic energy of a river 200 ft. wide and 15 ft. deep, if it flows at the rate of 4 mi. per hour, the weight of a cubic foot of water being 62.5 lb.? What horse power might be developed by using all the water in the river?

Problem 196. The flow of water in Niagara River is approximately 270,000 cu. ft. per second. What is its kinetic energy? What horse power could be developed by using all the water?

Problem 197. This amount of water, 270,000 cu. ft., goes over the falls of Niagara every second. The height of the falls including rapids above and below is 216 ft. What horse power could be developed by using all the water? What horse power could be developed by using the water, considering the height of the falls to be 165 ft., the height of free fall?

NOTE. It is estimated that the total horse power of Niagara Falls, considering the fall as 216 ft., is 7,500,000. The Niagara Falls Power Company diverts a part of the volume of water above the rapids into their power plants, where it passes through a tunnel into the river below the falls. The turbines are 140 ft. below the water level, and each one is acted upon by a column of water 7 ft. in diameter. The estimated power utilized in this way is 220,000 horse power. The student should estimate the horse power of each turbine, assuming the water to fall from rest through 140 ft. For a full account of the power at Niagara Falls, the student is referred to Proc. Inst. C. E., Vol. CXXIV, p. 223.

When the motion of the body is not along the line of the force, as is the case in Fig. 155, where the body is supposed moving up the plane under the system of forces shown, we resolve the force into components along and perpendicular to the direction of motion. It is evident that the component perpendicular to the direction of motion can do no work in moving the body up or down the plane.

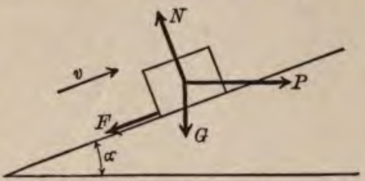


FIG. 155

The same might be said of the component of G perpendicular to the plane and of N, so that the work-energy equation in such a case includes only the components of the forces along the line of motion. The accelerating force in this case is $P \cos a$, and the resisting forces are F and $G \sin a$. The work-energy equation becomes

$$\frac{Mv^2}{2} - \frac{Mv^2_0}{2} + Fs + (G \sin \alpha)s = (P \cos \alpha)s,$$

where s is a distance measured along the plane.

Problem 198. A body whose weight is 32.2 lb. is pulled up an inclined plane, inclined at an angle of 30° with the horizontal, by a horizontal force of 250 lb. The motion is resisted by a constant force of friction of 10 lb. acting along the plane. If it starts from rest, what will be its velocity after it has gone up a distance of 100 ft.?

Problem 199. The same body as that in the preceding problem is projected down the plane with a velocity of 5 ft. per second. How far will it go before coming to rest?

HINT. In this case there is no working force acting, and the final kinetic energy is zero, so that the work-energy equation reduces to the expression: the work of resistance equals the initial kinetic energy.

Problem 200. The student should solve Problem 112 by using the principle of work and energy.

136. Work under the Action of a Variable Force. — When the forces are not constant, the work-energy equation already derived does not hold true. In such a case the equation $vdv = ads$ becomes, by integrating,

$$\frac{Mv^2}{2} - \frac{Mv_0^2}{2} + \int_0^s Rds = \int_0^s Pds.$$

The integrals expressed cannot be determined until it is known how R and P vary. In any case, however, we see that the quantity under the integral sign represents work, and so we may say: the work done equals the resistance overcome plus the change in kinetic energy.

Let us suppose that the resistance R varies as the distance, and also that the force P varies as the distance. Then $R = \text{const.} \times (s) = C_1 s$ and $P = \text{const.} \times (s) = C_2 s$. The work-energy equation becomes, upon substitution,

$$\frac{Mv^2}{2} - \frac{Mv_0^2}{2} + C_1 \int_0^s sds = C_2 \int_0^s sds.$$

In Art. 81 the case of a body of 644-lb. weight falling in a resisting medium was discussed. We may now discuss this same problem by means of the principle of work and energy. In this case,

$R = 10\,s$, $P = G$, $M = 20$, $v_0 = \sqrt{2gh} = 62.1$ ft. per second.

Then $$\frac{20\,v^2}{2} - \frac{20\,(62.1)^2}{2} + 10\int_0^s sds = 644 \int_0^s ds,$$

$$v^2 = 3864 - \frac{1}{2}s^2 + 64.4\,s.$$

This gives a relation between velocity and distance. The student should complete the problem, as outlined in Art. 81.

Problem 201. A body whose weight is 64.4 lb. falls freely from rest from a height of 5 ft. upon a 200-lb. helical spring. Find the compression in the spring.

It will be recalled from Problem 113 that a 200-lb. spring is such a spring as would be compressed 1 in. by a weight of 200 lb. resting upon it. It will also be recalled that in compressing such a spring, the resistance of the spring is at first zero, and that it increases in proportion to the compression. So in the present case we may write $\frac{200}{\frac{1}{12}} = \frac{R}{s}$, where the distance of one inch is expressed in feet; since s is expressed in feet and R is the resistance of the spring in pounds, $R = 2400\,s$. In this case $P = G, v = 0, v_0 = \sqrt{2\,gh} = 17.9$ ft. per second, and $M = 2$. Then the work-energy equation gives

$$-\frac{2\,(17.9)^2}{2} + 2400\,\frac{s^2}{2} = 64.4\,s,$$

$$s = .544 \text{ ft., or } 6.52 \text{ in.}$$

Problem 202. A weight of 500 lb. is to fall freely from rest through a distance of 6 ft. The kinetic energy is to be absorbed by a helical spring. Specifications require that the spring shall not be compressed more than 2 in. Find the strength of the spring required.

Problem 203. It requires 2000 lb. to press a certain sized nail into a board a distance of 2 in. The same size nail is to be driven to the same depth by a 5-lb. hammer (Fig. 156) in 4 blows. With what velocity must the hammer strike the nail each time? Assume that all the energy of the hammer is absorbed by the nail and that the resistance offered by the timber in question varies as the distance of penetration of the nail. Neglect the weight of the hammer as a working force. Under the assumptions made, the penetration of the nail will be the same for each blow.

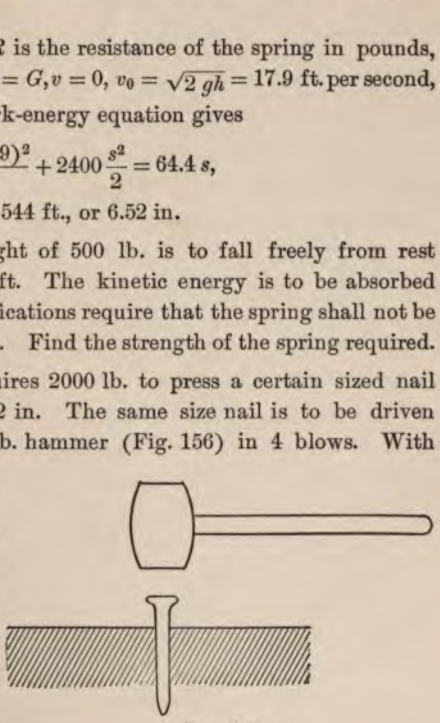


FIG. 156

Problem 204. Specifications state that it shall require 32,000 lb. to compress a helical spring 1½ in. What weight falling freely from rest through a height of 10 ft. will compress it one inch?

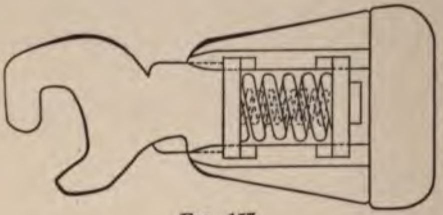


FIG. 157

Problem 205. The draft rigging of a freight car shown in Fig. 157 is provided with two helical springs, one inside the other. The outside spring is a 10,000-lb. spring, and the inside a 5000-lb. spring. A car weighing 60,000 lb. is provided with such a draft rigging. While going at the rate of 1 mi. per hour it collides with a bumping post. How much will the springs be compressed?

Problem 206. The draft rigging in the preceding problem is attached to the first car of a freight train, consisting of 30 cars, each weighing 60,000 lb. How much will the springs of the first car be elongated if there is 10 lb. pull for each ton of weight when the speed is 40 mi. per hour? The speed is increased to 45 mi. per hour. How much will the springs be elongated if the resistance per ton at this speed is 12 lb.?

Problem 207. The Mallet compound locomotive (*Railway Age*, Aug. 9, 1907) is capable of exerting a draw-bar pull of 94,800 lb. According to the preceding problem, how many 60,000-lb. cars can be pulled at 45 mi. per hour? What strength of spring would be necessary for the first car, if the allowable compression is 1½ in.?

Problem 208. An automobile going at the speed of 30 mi. per hour comes to the foot of a hill. The power is then shut off and the machine allowed to "coast" up hill. If the slope of the hill is 1 ft. in 50, how far up the hill will it go, if friction acting down the plane is .06 G, where G is the weight of the machine?

137. Pile Driver.—A pile driver consists essentially of a hammer of weight G so mounted that it may have a free fall from rest upon the pile (Fig. 158). The safe load to be placed upon a pile after it has been driven is the problem that interests the engineer. This is usually determined by driving the pile until it sinks only a certain fraction of an inch under each blow, then the safe load is a fraction of the resistance offered by the earth to these last blows. This resistance is very small when the pile begins to penetrate the earth, but increases as penetration proceeds, until finally due to the last blows it is nearly constant. If we regard the hammer G as a freely falling body, and consider the hammer and pile as rigid bodies, and further assume, as is usually done, that R for the last few blows is constant, we may write the work-energy equation,

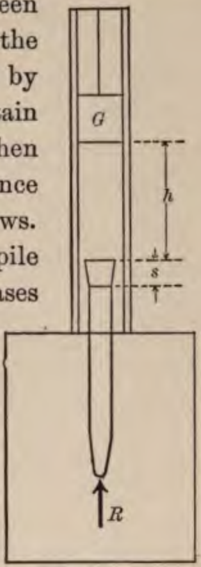


FIG. 158

$$\tfrac{1}{2} Mv_0^2 = \int_0^s R\,ds = Rs_1,$$

since the final kinetic energy is zero and the weight of the hammer as a working force is negligible. The distance s_1 is the amount of penetration of the pile for the blow in question. But $v_0^2 = 2gh$, so that

$$\tfrac{1}{2} Mv_0^2 = Gh.$$

We have then as the value for the supporting power of a pile,

$$R = \frac{Gh}{s_1}.$$

A safe value, R' from $\frac{1}{6}$ to $\frac{1}{8}$ of R, is taken as the safe supporting power of piles. The factor of safety and the value of s_1 for the last blow are usually matters of specification in any particular work. This is the formula given by Weisbach and Molesworth. Other authorities give formulæ as follows:

Trautwine, $R = 60\ G\sqrt[3]{h}$, if s_1 is small,

and $$R = \frac{5\ G\sqrt[3]{h}}{s_1 + 1};$$

Wellington, $R = \dfrac{12\ Gh}{s_1 + 1}$, where h is in feet and s_1 in inches;

McAlpine, $R = 80[G + (.228\sqrt{h} - 1)2240]$;

Goodrich, $R = \dfrac{10\ Gh}{3\ s_1}$.

For other formulæ and a general discussion of the subject of the bearing power of piles, the reader is referred to Transactions of Am. Soc. C. Eng., Vol. 48, p. 180.

The great number of formulæ for the supporting power of piles is due to the various assumptions that are made in deriving them. In deriving the Molesworth formula, the hammer and pile were considered as rigid bodies. It will be seen that the hammer and pile are both elastic bodies, both are compressed by the blow; there is friction of the hammer with the guides, and the cable attached to the hammer runs back over a hoisting drum. There is, in most cases, a loss of energy due to brooming of the head of the pile. This broomed portion must be cut off before noting the penetration due to the last few blows.

Results of tests have also been taken into consideration and have modified the formulæ. The Wellington formula differs from the Molesworth formula only in the denominator, where $s_1 + 1$ is used instead of s_1. This has been done to guard against the very large values of R given when s_1 is very small. According to the Wellington formula, R can never be greater than Gh, the total energy of the hammer, and this is perhaps the safer formula to use for small values of s_1.

Engineers have come to believe that it will be extremely difficult to get a general formula that will give very exact information as to the bearing power of piles, since soil conditions are so varied. The more simple formulæ with a proper factor of safety are used.

As an illustration of the use of the formula, let us consider the problem of providing a pile to support 75 tons. If the weight of the hammer is 3000 lb., and the height of fall 15 ft., the pile will be considered *down* when

$$s_1 = \frac{Gh}{R} = \frac{3000 \times 15}{150,000} = .3 \text{ ft.} = 3.6 \text{ in.}$$

Using a factor of safety of $\frac{1}{6}$, we have $s_1 = .6$ in.

Problem 209. Compute the value of s_1 for the pile in the above illustration by using the various formulæ given in this article. Compare the results.

Problem 210. A pile is driven by a 4000-lb. hammer falling freely 20 ft. What will be the safe load that the pile will carry if at the last blow the amount of penetration was $\frac{1}{8}$ in.? Use a factor of safety of $\frac{1}{4}$. Compute by the Molesworth and the Wellington formulæ, and compare.

Problem 211. A pile was driven by a steam hammer. The last twenty blows showed a penetration of one inch. If two blows of the steam hammer cause the same penetration as one blow from a 2000-lb. hammer falling 20 ft., what weight in tons will the pile support? Assume the penetration for each of the last few blows the same.

138. Steam Hammer. — The steam hammer consists essentially of a steam cylinder mounted vertically and having a weight or hammer attached to one end of the piston rod. Let *AB* (Fig. 159) be the steam cylinder, *D* the piston, and *H* the anvil, upon which a piece of metal is shown under the hammer *G*. The steam pressure in the cylinder is constant and equal to *P*, while the piston passes over a distance *a* to cut-off, and varies inversely as the volume over the remaining distance *b*. *G*, the weight of the hammer and piston, is also a working force. The resistance, *R′*, of the metal varies during any blow with the amount of compression. It is zero just as the hammer touches the metal, and increases up to a maximum when the compression is greatest. Let *R′* be the average resistance of the metal, and R_1 the exhaust pressure. Then the work-energy equation for the hammer before striking the metal becomes

$$\frac{Mv^2}{2} - \frac{Mv_0^2}{2} + R_1\int_0^s ds = P\int_0^a ds + \int_a^s P'ds + G\int_0^s ds,$$

or

$$\frac{Mv^2}{2} - \frac{Mv_0^2}{2} + R_1 s = Pa + Gs + \int_a^s P'ds.$$

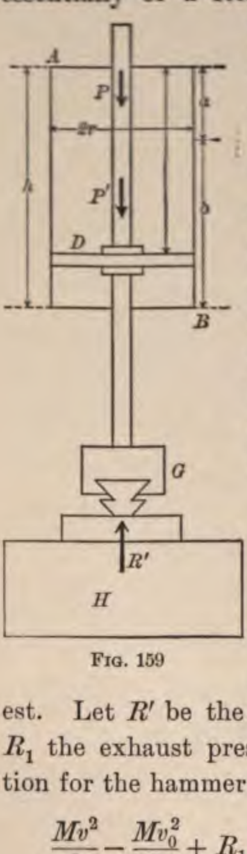
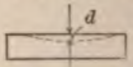


FIG. 159



Since P' varies inversely as the volume of the cylinder, we may write, $P' = \frac{\text{const.}}{\pi r^2 s} = \frac{c}{s}$.

Then the work-energy equation gives

$$\frac{Mv^2}{2} + R_1 s = Pa + Gs + c \log \frac{s}{a}.$$

The term $\frac{Mv_0^2}{2}$ is zero, since the motion has been considered from rest at the top of the cylinder to a distance s. The quantity c may be computed by reading from the indicator card the value of P' at s. It will be seen that s has been taken greater than a; that is, the piston is beyond the point of cut-off.

When the hammer finally comes to the face of the metal, the work-energy equation may be written

$$\frac{Mv_1^2}{2} + R_1 h' = Pa + Gh' + c' \log \frac{b'}{a},$$

where the distance h' represents the value of s when the hammer just touches the metal, and v_1, b', and c' are the corresponding values of v, b, and c. This equation gives the kinetic energy of the hammer when it strikes the metal. The work-energy equation for the hammer during the compression of the piece may now be written

$$\frac{Mv_1^2}{2} = R' d,$$

where d is the amount of compression of the metal due to the blow. This is shown by the small figure to the right, where the piece of metal has been drawn to a somewhat larger scale.

After the hammer strikes the metal, the steam pressure and the weight of the hammer as working forces, and the exhaust pressure as a resisting force, have been neglected. The work done by these pressures is small, since the distance d is small. Approximately, then, the work done on the metal equals the kinetic energy at the time of first contact.

Instead of using the value $P' = \frac{c}{s}$, and computing the integral $\int P'ds$ as indicated in the formulæ, values of P' and s might be read from the indicator diagram (see Art. 131) and added by means of Simpson's formula (see Art. 26).

As an illustration of the foregoing, let us suppose the steam cylinder 25 in. long and 14 in. in diameter; the steam pressure $P = 18{,}000$ lb.; the exhaust pressure $R_1 = 2300$ lb.; $a = 7.2$ in.; $d = \frac{1}{4}$ in.; $G = 644$ lb.; $c' = 10{,}800$ lb.; $h' = 24$ in. Substituting in the work-energy equation, we have for the kinetic energy of the hammer at the time of striking the iron,

$$\frac{Mv_1^2}{2} = 11{,}475 \text{ ft.-lb.}$$

This gives a value for $v_1 = 33.8$ ft. per second as compared with 11.3 ft. per second for the same weight freely falling through the same distance.

Investigating now the resistance of the metal, we have, under the assumption already made,

$$R'd = 11{,}475 \text{ ft.-lb.},$$

so that

$$R' = 550{,}800 \text{ lb.}$$

In the above discussion we have neglected the compression of the anvil and hammer due to the blow, and also the friction of the piston.

Problem 212. Find the kinetic energy of the hammer when $h' = 18$ in. Find also v and R', using the same value of d.

Problem 213. A steam hammer exactly similar to the one given in the illustration above is used with the same steam pressure. It is only necessary, however, for the work for which it is intended, that the kinetic energy of the hammer for a stroke of 2 ft. be 6000 ft.-lb. What weight of hammer should be used?

Problem 214. Compute the kinetic energy and velocity of the hammer in the illustration ($G = 644$ lb.) when the piston has moved the full length of the cylinder ($h' = 25$ in.). Assume that there is nothing on the anvil.

Problem 215. What value of h' in the above problem would give the hammer the same velocity as it would have if it fell freely from rest through the height h? Compute the kinetic energy for this velocity.

Problem 216. In the illustration given above, what would be the value of R' if the steam pressure and G be included as working forces, and R_1 as a resisting force, during the compression of the piece?

Problem 217. In the illustration given above, suppose that, in addition to the compression of the piece $\frac{1}{4}$ in., the anvil is compressed .02 in. Find the value of R'.

139. Energy of Rotation about Fixed Axis. — In Art. 103, where the subject of the rotation of a rigid body about a fixed axis was discussed, the following equation was derived:

$$\Sigma(P'_1 d_1 + P'_2 d_2 + P'_3 d_3 + \text{etc.}) = \theta I,$$

where the P's represent forces tending to rotate or retard the rotation of a rigid body about a fixed axis, the d's, the distances of the lines of action of these forces from O, θ the angular acceleration, and I the moment of inertia of the body with respect to the axis of rotation.

This equation may be put in the form

$$\theta = \frac{\Sigma\,(P'_1d_1 + P'_2d_2 + P'_3d_3 + \text{etc.})}{I}.$$

Now let us suppose the moment P'_1d_1 is made up of a working moment and a resisting moment, such that $P'_1d_1 = P''_1d_1 - R''_1d''_1$, $P'_2d_2 = P''_2d_2 - R''_2d''_2$, etc. Remembering that $\omega d\omega = \theta d\alpha''$, we may write, after clearing of fractions,

$$I\omega d\omega = \Sigma(P''_1d_1d\alpha'' + P''_2d_2d\alpha'' + P''_3d_3d\alpha'' + \text{etc.}$$
$$- \Sigma(R''_1d''_1d\alpha'' + R''_2d''_2d\alpha'' + R''_3d''_3d\alpha'' + \text{etc.}).$$

Let the angles which P''_1, P''_2, P''_3, etc., make with the x-axis be called α''_1, α''_2, α''_3, etc. Then $d_1d\alpha'' = ds_1$, $d_2d\alpha'' = ds_2$, $d_3d\alpha'' = ds_3$, and $d''_1d\alpha'' = ds''_1$, etc. Then if continuity exists so that we may integrate, we may write

$$I\int_{\omega_0}^{\omega}\omega d\omega = \int_0^s P''_1ds_1 + \int_0^s P''_2ds_2 + \text{etc.}$$
$$-\int_0^s R''_1ds''_1 - \int_0^s R''_2ds''_2 - \text{etc.},$$

or

$$I\left(\frac{\omega^2 - \omega_0^2}{2}\right) = \int_0^s P''_1ds_1 + \int_0^s P''_2ds_2 + \text{etc.}$$
$$-\int_0^s R''_1ds''_1 - \int_0^s R''_2ds''_2 - \text{etc.}$$

Since $\frac{1}{2}\,\omega^2 I = \frac{1}{2}\,\omega^2\int dM\rho^2 = \int\frac{1}{2}\,dM(\omega\rho)^2 = \int\frac{1}{2}\,dMv^2,$

the kinetic energy of the body, where ω is the angular velocity, the left-hand side represents the difference in the kinetic energy of rotation of the body when its initial velocity is ω_0 and its final velocity ω. On the right-hand side, Pds represents work, since ds is measured along the line of action of P in each case. A similar statement could be made for the R's, so that the right-hand side represents the work of the working forces minus the work of the resisting forces. Here, then, as in simple translation, between any two positions of a rigid body, *the work done by the working forces equals the work done by the resisting forces plus the change in kinetic energy.*

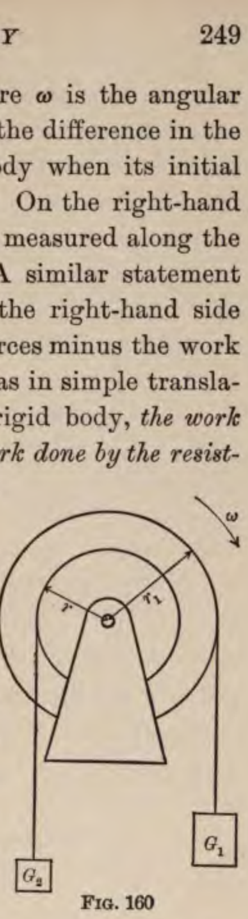


FIG. 160

As an illustration, let us consider the case of two weights (see Fig. 160), $G_1 = 20$ lb. and $G_2 = 10$ lb., suspended from drums rigidly attached to each other and of radii 3 ft. and 2 ft. respectively. Let the weight of the two drums and shaft be 644 lb., and the radius of gyration 2 ft. The radius of the axle is one inch and the axle friction 30 lb. The friction acts tangentially to the axle.

Assume that the initial velocity ω_0 is one radian per second, and the final velocity 18 radians per second, how many revolutions will the drums make?

The work-energy equation gives

$$\frac{1}{2}(18)^2(80) - \frac{1}{2}(1)^2(80) + G_2 2\pi r n + 30\frac{2\pi r_3}{12}n = G_1 2\pi r_1 n,$$

where r_1 is the radius of the large drum, r that of the small drum, r_3 that of the axle, and n the number of revolutions. Making the substitutions, n becomes

$$n = 54.8 \text{ revolutions.}$$

Problem 218. In the above illustration, what is the velocity of G_1 and G_2 when ω has its initial and final values? In what time do the drums make the 54.8 revolutions?

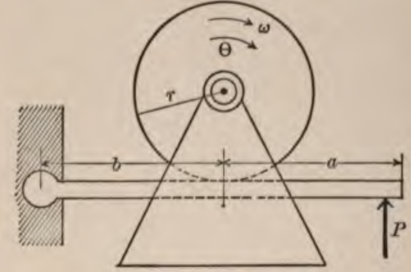


FIG. 161

Problem 219. The drum in Fig. 161 is solid and has a radius r and a thickness h. Initially, it is rotating, making ω_0 radians per second, but it is brought to rest by the action of a brake. The brake is applied from below by a force P acting at the end of the beam. The force of friction between the drum and brake is $\frac{P'}{4}$, where P' is the normal pressure exerted by the beam on the drum. The radius of the axle is r_1, and the axle friction (.05) P'', where P'' is the pressure of the axle on the bearing (neglecting the lifting caused by P). Required the work-energy equation.

Since the drum comes to rest, the final kinetic energy is zero, so that

$$-\frac{1}{2}\omega_0^2 I + \frac{P'}{4} 2\pi r n + (.05) P'' 2\pi r_1 n = 0.$$

There are no working forces, so we find the equation reducing to the form: the initial kinetic energy equals the work of resistance. The normal pressure exerted by the beam on the drum may be found by taking moments about the hinge of the beam. Then

$$P' = \frac{a+b}{b} P.$$

The number of revolutions turned through in coming to rest is designated by n. The equation then becomes

$$\frac{1}{2}\,\omega_0^2\,I = \frac{\pi r n\,(a+b)\,P}{2\,b} + (.05)\,P''\,2\,\pi r_1 n.$$

Problem 220. Suppose the drum in the preceding problem to be 3 ft. in diameter, $1\frac{1}{3}$ in. thick, and made of cast iron. It is making 4 revolutions per second when the force $P = 100$ lb. is applied to the beam. The length of the drum is 6 ft., and the rim weighs twice as much as the spokes and hub. If $k = 1.25$ ft., $a + b = 8$ ft., and $r_1 = 1$ in., find the number of revolutions n_1 that the drum will make before coming to rest. Assume the friction of the brake on the drum to be $\frac{1}{2}$ the normal pressure, and the friction of the axle $(.05)\,P''$.

Problem 221. The drum in the preceding problem is making 3 revolutions per second; what force will be required to bring it to rest in 100 revolutions?

Problem 222. If the brake in Problem 220 is above instead of below the drum, how will the results in Problems 220 and 221 be changed?

Problem 223. A square prism as shown in Fig. 162 is mounted so as to rotate due to the weight G. The elastic cord runs over the pulley B and meets the square at P'. The mechanism is such that motion begins when P is in the position shown, and ceases when the prism has made a quarter turn; that is, when P reaches P'. The diameter of the journal is 2 in., and the weight on the same is 600 lb. The force of friction on the journals is 60 lb., and on the pulley at B 10 lb. Find the tension in the cord when P reaches P'. The cord is elastic, and is made of such material that it elongates, due to a pull of 100 lb., .02 in. in each inch of length. What is the elongation per inch due to the fall of G as stated?

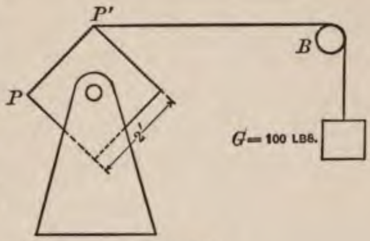


FIG. 162

140. Brake Shoe Testing Machine. — The brake shoe testing machine owned by the Master Car Builders' Association has been established at Purdue University. It consists of a heavy fly wheel attached to the same axle as the car wheel. These are connected with the engine, and may be given any desired rotation. When this has been obtained, they may be disconnected and allowed to rotate. The dimensions and weight of the parts are known so that the kinetic energy of the fly wheel and rotating parts may be computed by noting the angular velocity. When the desired velocity has been attained, the brake shoe is brought down on the car wheel. The required normal pressure on the shoe at *A* (see Fig. 163) is obtained by applying suitable weights at *B*. The system of levers is such that one pound at *B* gives a normal pressure of 24 lb. on the brake shoe. The weight of the levers themselves gives a normal pressure of 1233 lb. Provision is also made for measuring the tangential pull of the brake friction; this, however, is not shown in the figure.

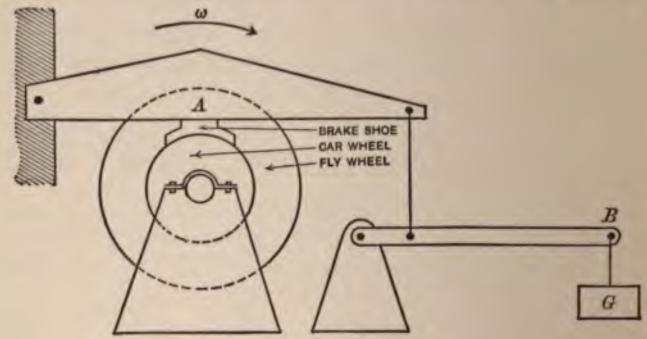


FIG. 163

The weight of the fly wheel, car wheel, and shaft and all rotating parts is 12,600 lb., and the radius of gyration is $\sqrt{2.16}$. The weight of 12,600 lb. is supposed to be the greatest weight that any bearing in passenger or freight service will be called upon to carry. The diameter of the fly wheel is 48 in., its thickness 30 in., diameter of shaft 7 in., and the diameter of the car wheel is 33 in. The brake-shoe friction is $\frac{1}{4}$ the normal pressure of the brake shoe on the wheel, and the journal friction may be assumed as (.002) of the pressure of the axle on the bearing. The work-energy equation for the rotating parts after being disconnected from the engine becomes

$$\frac{1}{2}\left(\frac{12{,}600}{32.2}\right)\omega^2 - \frac{1}{2}\left(\frac{12{,}600}{32.2}\right)\omega_0^2 + (1233 + 24\ G)\frac{1}{4}\,2\,\pi\frac{33}{24}\,n$$
$$+ (1233 + 12{,}600 + 24\ G)(.002)2\,\pi\,\frac{7}{24}\,n = 0,$$

since there are no working forces.

Problem 224. The speed is such as to correspond to a speed of train of a mile a minute when brakes are applied. What must be the weight G so that a stop may be made in a thousand feet? What is the corresponding normal pressure on the brake shoe?

Problem 225. If the speed corresponds to the speed of a train of 100 mi. per hour, what weight G would be necessary to reduce the speed to 60 mi. per hour in one mile? What is the normal pressure on the brake shoe necessary?

Problem 226. If the velocity corresponds to a train velocity of 60 mi. per hour, and the apparatus is brought to rest in 220 revolutions, the weight G is 100 lb. Find the tangential force of friction acting on the face of the wheel. What relation does this bear to the normal brake-shoe pressure?

NOTE. In the preceding problems, the ratio (the coefficient of friction, see Art. 146) has been taken as $\frac{1}{4}$. One of the important uses of this testing machine is to determine the coefficient of friction for different types of brake shoes. Experiment shows that it varies generally from $\frac{1}{5}$ to $\frac{1}{4}$, sometimes going as high as $\frac{3}{10}$.

141. Work of Combined Rotation and Translation. — The relation between work and energy of simple translation and the work and energy of rotation about a fixed axis have been discussed. We shall now determine the relation for combined rotation and translation when the axis remains parallel to itself. Let the body of mass M (Fig. 164) be in rotation with angular velocity ω about an axis at O, and at the same time let this axis move parallel to itself with a linear velocity v. At any instant the elementary mass dM has a linear velocity of translation v_1 and a tangential velocity $v = \omega\rho$. Its resultant velocity is expressed as the diagonal of a parallelogram constructed upon the two velocity arrows as sides, so that

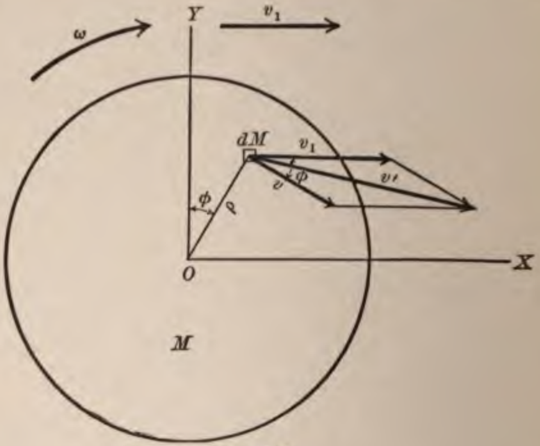


FIG. 164

$$v'^2 = v^2 + v_1^2 + 2\,vv_1 \cos\phi.$$

Multiplying both sides of this equation by $\frac{dM}{2}$, we have

$$\int \frac{dMv'^2}{2} = \int \frac{dMv^2}{2} + \int \frac{dMv_1^2}{2} + \int 2\frac{dM}{2} vv_1 \cos\phi,$$

or $$\int \frac{dMv'^2}{2} = \int \frac{dM}{2}\omega^2\rho^2 + \int \frac{dMv_1^2}{2} + \int dM\omega\rho v_1 \cos\phi;$$

but $\rho \cos\phi = y$, and $\int dMy = M\bar{y} = 0$, since OX is a gravity line. At any instant ω and v_1 are constant. Therefore,

$$\text{K.E.} = \int \frac{dMv'^2}{2} = \frac{1}{2}\omega^2 I + \frac{1}{2} Mv_1^2.$$

At any instant, then, *the kinetic energy of combined rotation and translation is equal to the kinetic energy of translation of the center of gravity plus the kinetic energy of rotation.*

As an illustration, consider a disk of radius r and thickness h rolling without slipping down an inclined plane, inclined at an angle α with the horizontal (see Fig. 165). There is a working force $G \sin\alpha$ and a resisting force $F = (.06)$ $G \cos\alpha$. Now the kinetic energy of the disk is made up of the sum of its kinetic energy rotation and translation. If ω_0 and v_0 be the respective initial angular and linear velocities, we have

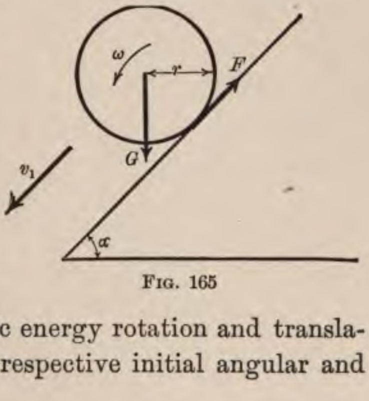


FIG. 165

$$\tfrac{1}{2}\omega^2 I - \tfrac{1}{2}\omega_0^2 I + \tfrac{1}{2} Mv_1^2 - \tfrac{1}{2} Mv_0^2 + Fs = G \sin\alpha \cdot s.$$

Problem 227. Suppose the disk in the above illustration to be made of cast iron, and let $r = 2$ ft., $h = \frac{1}{2}$ ft., and $\alpha = 30°$. At a certain instant it is making 2 revolutions per second. What will be the linear and angular velocities after the disk has gone 20 ft.? Would the disk finally come to rest?

142. Kinetic Energy of Rolling Bodies. — It is convenient to express the kinetic energy of combined rotation and translation of such bodies as rolling wheels in a different form from that given in the preceding article. There is some mass M_1 that will have the same kinetic energy when translated with a velocity v_1 as the kinetic energy of translation plus the kinetic energy of rotation of the body of mass M; that is,

$$\frac{M_1 v_1^2}{2} = \frac{M v_1^2}{2} + \frac{\omega^2 I}{2},$$

for a wheel rolling on a straight track $\omega r = v_1$, where r is the radius.

Then $$M_1 = M + \frac{I}{r^2}.$$

This has been called the equivalent mass.

Applying this to the disk in the preceding article, we find the work-energy equation to be,

$$\frac{M_1 v_1^2}{2} - \frac{M_1 v_0^2}{2} + Fs = G \sin \alpha \cdot s$$

for the disk, since $I = \frac{1}{2} M r^2$, $M_1 = \frac{3}{2} M$.

Problem 228. A sphere of radius r rolls without slipping down an inclined plane, inclined at an angle α to the horizontal, with an initial velocity v_0. Show that its kinetic energy is the same as that of a sphere whose mass is $\frac{2}{5}$ larger translated with a velocity v_1.

Problem 229. The sphere in the preceding problem is made of steel, 12 in. in diameter, and $\alpha = 30°$. If $v_0 = 10$ ft. per second, what will be the velocity 10 ft. down the plane? There is a force of friction acting up the plane = (.03) times the normal pressure of the sphere on the plane.

143. Work-Energy Relation for Any Motion. — The relation between work and energy for the motions considered in this chapter holds for more complicated motions and for motions in general. The limits of the present work will not admit the proof of the general theorem. It may be said, however, that for any motion the work done by the working forces equals the work done by the resisting forces plus the change in kinetic energy. In the case of the motion of a complicated machine, the total work done equals the total resistance overcome plus the change in kinetic energy of the various parts of the machine.

144. Work done when Motion is Uniform. — When the motion is uniform, the change in kinetic energy is zero, and the work-energy equation reduces to the form: *work done equals the resistance overcome.*

As an illustration, let us consider the case of a locomotive moving at uniform speed and represented in Fig. 166. Suppose P the mean effective steam pressure (see Art. 131), F the friction of the piston, F' the friction of the crosshead, F'' the journal friction, F''' the crank-pin friction, T the friction on the rail, R the draw bar resistance, G the weight of the locomotive, and N' and N the normal reactions of the rails on the wheels. Consider only one side of the locomotive and write the work-energy equation for a distance s, equal to a half turn of the driver (from dead center A to dead center B), that the locomotive travels. This becomes, for the frame,

$$P\pi a = F(\pi a + 2r) + F'(\pi a + 2r) + R'\pi a - R\pi a,$$

where R' is the pressure of the driver axle on the frame. If we neglect friction, this becomes

$$\boldsymbol{P = R' - R.}$$

Considering the rotating and oscillating parts, we obtain

$$P(\pi a + 2r) = F(\pi a + 2r) + F'(\pi a + 2r) + T\pi a \\ + F''\pi r_1 + F'''\pi r_2 + R'\pi a,$$

where r_1 and r_2 are the radii of the driver axle and the crank pin, respectively. If we neglect friction, this equation reduces to the form,

$$P(\pi a + 2r) = T\pi a + R'\pi a.$$

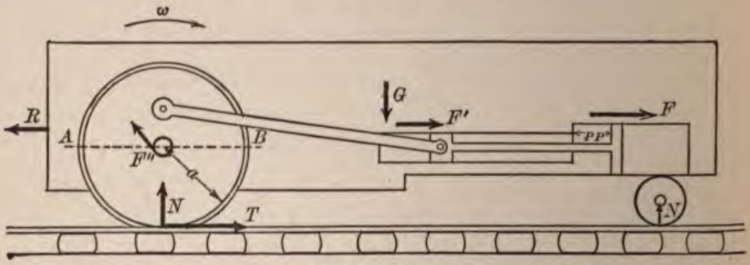


Fig. 166

Taking all the parts represented in Fig. 166, we may disregard the work of friction of the cylinder and crosshead, since the sum would be zero. That is, the work done by the piston on the cylinder, due to friction, equals the work done by the cylinder on the piston due to friction. We have, then, for the work-energy equation,

$$P(\pi a + 2r) = P\pi a + R\pi a + T\pi a + F''\pi r_1 + F'''\pi r_2.$$

It is seen that the pressure of the steam on the head of the cylinder, for the half of the stroke considered, is a resistance. If we neglect friction and assume perfect rolling, this equation becomes

$$P(\pi a + 2r) = P\pi a + R\pi a,$$

or

$$\boldsymbol{P = \frac{\pi a}{2r} R},$$

or considering both cylinders,

$$P = \frac{\pi a}{4r} R.$$

This is the formula usually given for the tractive power of a locomotive having single expansion engines. This may be expressed in terms of the dimensions of the cylinders and the unit steam pressure. Let p be the unit steam pressure in pounds per square inch, l the length of the cylinder in inches, d the diameter of the cylinder in inches, and d_1 the diameters of the drivers in inches; then

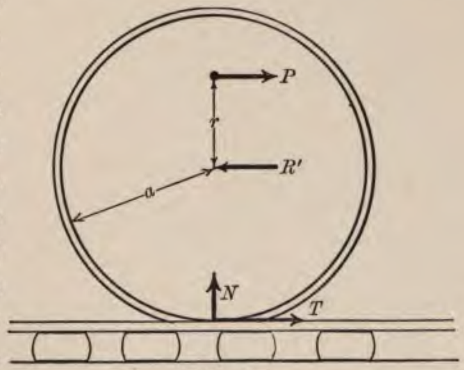


FIG. 167

$$R = \frac{d^2pl}{d_1}.$$

Considering the forces acting on the driver, disregarding friction, and taking moments about the center of the wheel (see Fig. 167), we have, for uniform motions,

$$Pr = Ta,$$

or

$$T = \frac{r}{a} P.$$

Taking moments about the point of contact of the wheel and rail, we have, for the position shown,

$$P(a + r) = R'a,$$

and since

$$P = R' - R,$$

we have

$$R = \frac{r}{a} P.$$

It follows that $T = R$; that is, the train resistance cannot be greater than the adhesion of the drivers to the rails.

This adhesion in American practice is usually taken as $\frac{1}{4}$ to $\frac{1}{5}$ the load on the drivers.

Problem 230. What resistance R may be overcome by a locomotive moving at uniform speed, diameter of drivers 62 in., cylinders 16 × 24 in., and a steam pressure on the piston of 160 lb. per square inch? What should be the weight of the locomotive on the drivers?

Problem 231. If the diameter of the drivers of a locomotive is 68 in., and the size of the cylinder is 20 × 24 in., what train resistance may be overcome by a steam pressure of 160 lb. per square inch?

Problem 232. A locomotive has a weight of 155 tons on the drivers, if the adhesion is taken as $\frac{1}{5}$, this allows 31 tons for the draw-bar pull. The train resistance per ton of 2000 lb., for a speed of 60 mi. per hour, is 20 lb. Find the weight of the train that can be pulled by the locomotive at the speed of 60 mi. per hour.

Problem 233. An 80-car freight train is to be pulled by a single expansion locomotive at the rate of 30 mi. per hour. The weight of each car is 60,000 lb., and the resistance for this speed is 10 lb. per ton. What must be the weight on the drivers, if the adhesion is $\frac{1}{5}$?

CHAPTER XIV

FRICTION

145. Friction. — When one body is made to slide over another, there is considerable resistance offered because of the roughness of the two bodies. A book drawn across the top of a table is resisted by the roughness of the two bodies. The rough parts of the book sink into the rough parts of the table so that when one of the bodies tends to move over the other, the projections interfere and tend to stop the motion. The bearings of machines are made very smooth, and usually we do not think of such surfaces as having projections. Nevertheless they are not perfectly smooth, and when one surface is rubbed over the other, resistance must be overcome. This resisting force to the motion of one body over another is known as *friction*. When the bodies are at rest relative to each other, the friction is known as the *friction of rest*, or *static friction*. When they are in motion with respect to each other, the friction is known as the *friction of motion*, or *kinetic friction*.

146. Coefficient of Friction. — If the body represented in Fig. 168 be pulled along the horizontal plane by the force P, the following forces will

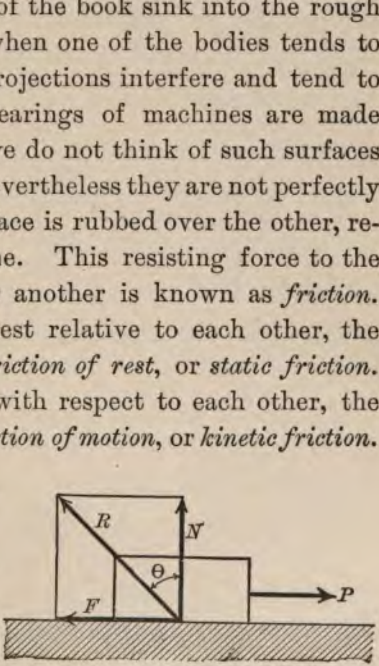


FIG. 168

be acting on it; the downward force G (not shown) and the reaction R inclined back of the vertical through the angle θ. The reaction R of the plane on the body may be resolved into two components, one horizontal and one vertical. The horizontal force is known as the force of friction, and the normal force, the normal pressure. The tangent of the angle θ, or $\frac{F}{N}$, is called the *coefficient of friction.* This coefficient of friction, which we shall represent by f, may be defined as *the ratio of the force of friction to the normal pressure; it is an absolute number.*

The coefficient of friction is usually determined by allowing a body to slide down an inclined plane as shown in Fig. 169. The angle θ is increased until the force of friction F will just keep the body from sliding down the plane. The angle θ is then called the *angle of repose*, and the tangent of θ is the coefficient of friction.

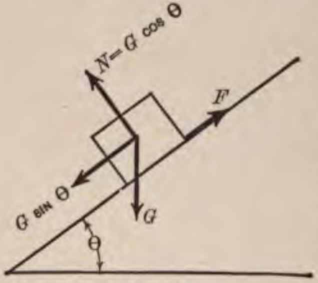


FIG. 169

It is possible with such an apparatus to determine the coefficient of friction for various materials. It has been found that after motion begins the friction is less, that is, *the friction of motion is less than the friction of rest.* This is an important law for engineers.

147. Laws of Friction for Dry Surfaces.—Very little was known of the laws of friction until within the last seventy-five years. About 1820 experiments were made that seemed to show that, for such materials as wood, metals, etc., friction varies with the pressure, and is independent of the extent of the rubbing surfaces, the time of contact, and the velocity. A little later (1831) Morin published the following three laws as a result of his experiments on friction:

(1) *The friction between two bodies is directly proportional to the pressure; that is, the coefficient of friction is constant for all pressures.*

(2) *The coefficient and amount of friction for any given pressure is independent of the area of contact.*

(3) *The coefficient of friction is independent of the velocity, although static friction is greater than kinetic friction.*

These laws of Morin hold approximately for dry unlubricated surfaces, although it has been found that an increase in speed lowers the coefficient of friction. The coefficient of friction is a little greater for light pressures upon large areas than for great pressures on small areas.

The following is a table of some of the coefficients of friction as determined by Morin:

COEFFICIENTS OF FRICTION, DUE TO MORIN

MATERIAL	CONDITION OF SURFACE	COEFFICIENT OF FRICTION	ANGLE OF FRICTION
Brick on limestone	Dry	.67	35° 50′
Cast iron on cast iron	Slightly greased	.16	9° 6′
Cast iron on oak	Wet	.65	30° 2′
Copper on oak		.17	9° 38′
Copper on oak	Greased	.11	6° 17′
Leather on cast iron		.28	15° 39′

COEFFICIENTS OF FRICTION, DUE TO MORIN — *Continued*

MATERIAL	CONDITION OF SURFACE	COEFFICIENT OF FRICTION	ANGLE OF FRICTION
Leather on cast iron	Wet	.38	20° 49′
Leather on cast iron	Oiled	.12	6° 51′
Leather on oak	Fibers parallel	.74	36° 30′
Leather on oak	Fibers crossed	.47	25° 11′
Oak on oak	Fibers parallel, dry	.62	31° 48′
Oak on oak	Fibers crossed, dry	.54	28° 22′
Oak on oak	Fibers parallel, soaped	.44	23° 45′
Oak on oak	Fibers crossed, wet	.71	35° 23′
Oak on oak	Fibers end to side, dry	.43	23° 16′
Oak on oak	Fibers parallel, greased	.07	4° 6′
Oak on oak	Heavily loaded, greased	.15	8° 45′
Oak on pine	Fibers parallel	.67	33° 50′
Oak on limestone	Fibers on end	.63	32° 15′
Oak on hemp cord	Fibers parallel	.80	38° 40′
Pine on pine	Fibers parallel	.56	29° 15′
Pine on oak	Fibers parallel	.53	27° 56′
Wrought iron on oak	Wet	.62	31° 48′
Wrought iron on oak		.65	33° 2′
Wrought iron on wrought iron		.28	15° 39′
Wrought iron on cast iron		.19	10° 46′
Wrought iron on limestone		.49	26° 7′
Wood on metal	Greased	.10	6° 0′
Wood on smooth stone	Dry	.58	30° 7′
Wood on smooth earth	Dry	.33	18° 16′

148. Friction of Lubricated Surfaces. — The laws of friction as given by Morin and stated in the preceding article hold approximately for rubbing surfaces, when the surfaces are dry or nearly so; that is, for poorly lubricated surfaces. If, however, the surfaces are well lubricated so that the projections of one do not fit into the other, but are kept apart by a film or layer of the lubricant, the laws of Morin are not even approximately true. The study of the friction of lubricated surfaces, then, may be divided into two parts: (1) the study of poorly lubricated bearings, and (2) the study of well-lubricated bearings, the friction of which varies from $\frac{1}{6}$ to $\frac{1}{10}$ that of dry or poorly lubricated bearings.

Since the friction of poorly lubricated bearings is about the same as that of dry surfaces, we shall consider that the laws of Morin hold, and shall confine our attention to the friction of well-lubricated bearings. If the lubricant is an oil, the friction of the bearing is no longer due to one surface rubbing over the other, but to the friction between the bearing and the oil, and to the internal friction of the oil. That is, the oil adheres to the two surfaces and its own particles attract each other, and the motion of one of the surfaces with respect to the other changes the positions of the oil particles. It is to be expected then that the friction of an oiled bearing will depend upon the *viscosity of the oil*, upon the *thickness of the layer interposed between the surfaces*, and upon the *velocity and form of the bearing*.

The coefficient of friction is no longer constant, but varies with the temperature, velocity, and pressure. The variation of the coefficient of friction of a paraffine oil with temperature is shown in Fig. 170 when the pressure on the bearing is 33 lb. per square inch and a velocity of rubbing of 296 ft. per minute. It is seen that the coefficient of friction *decreases* with increase of temperature until a temperature of 80° F. is reached, when it increases rapidly.

This means that above this temperature the oil is so thin that it is squeezed out of the bearing, and the conditions of dry bearing are approached. The temperature at which oils show an increasing coefficient of friction is different for different oils, even at the same pressure and velocity. The curve in Fig. 170, however, may be regarded as typical of all oils when the pressure and velocity are constant.

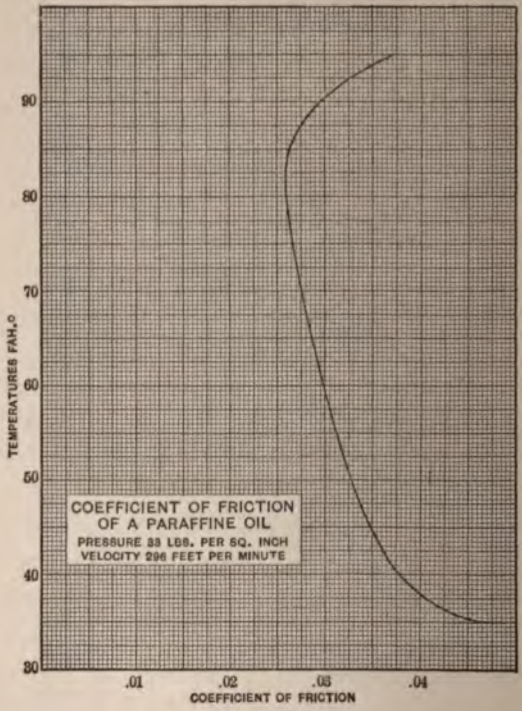


Fig. 170

The following table, due to Professor Thurston, shows the relation between the coefficient of friction and temperature for a sperm oil in steel bearings when the velocity of rubbing is 30 ft. per minute.

PRESSURE, LB. PER SQ. IN.	TEMPERATURE, DEGREES F.	COEFFICIENT OF FRICTION	PRESSURE, LB. PER SQ. IN.	TEMPERATURE, DEGREES F.	COEFFICIENT OF FRICTION
200	150	.0500	100	110	.0025
200	140	.0250	50	110	.0035
200	130	.0160	4	110	.0500
200	120	.0110	200	90	.0040
200	110	.0100	150	90	.0025
200	100	.0075	100	90	.0025
200	95	.0060	50	90	.0035
200	90	.0056	4	90	.0400
150	110	.0035			

It is seen that for a pressure of 200 lb. per square inch as the temperature increases from 90° F. the coefficient increases, indicating that the temperature of 90°, for the given pressure and velocity, was above the temperature at which the oil became so thin as to be squeezed out and the bearing to approach the condition of a dry bearing. For a constant temperature 110° F. and 90° F. the coefficient is seen to decrease with increase of pressure up to a certain point and then to increase. This is a typical behavior of oils when the temperature is constant and the pressure varies.

At speeds exceeding 100 ft. per minute, the same authority found "that the heating of the bearings within the above range of temperatures decreases the resistance due to friction, rapidly at first and then slowly, and gradually a temperature is reached at which increase takes place and progresses at a rapidly accelerating rate."

The relation between the coefficients of rest and of motion as determined by Professor Thurston for three oils is given below. The journals were cast iron, in steel boxes; velocity of rubbing 150 ft. per minute and a temperature 115° F.

	SPERM OIL			WEST VIRGINIA OIL			LARD		
PRESSURE, LB. PER. SQ. IN.	At 150 ft. per min.	At starting	At stopping	At 150 ft. per min.	At starting	At stopping	At 150 ft. per min.	At starting	At stopping
50	.013	.07	.03	.0213	.11	.025	.02	.07	.01
100	.008	.135	.025	.015	.135	.025	.0137	.11	.0225
250	.005	.14	.04	.009	.14	.026	.0085	.11	.016
500	.004	.15	.03	.00515	.15	.018	.00525	.10	.016
750	.0043	.185	.03	.005	.185	.0147	.0066	.12	.020
1000	.009	.18	.03	.010	.18	.017	.0125	.12	.019

Steel Journals and Brass Boxes.

500	.0025						.004		
1000	.008						.009		

It is seen that the coefficient of friction at starting is much greater than at stopping, and that these are both much greater than the value at a speed of 150 ft. per minute.

For an intermittent feed such as is given by one oil hole, without a cup, oiled occasionally, Professor Thurston found for steel shaft in bronze bearings, with a speed of rubbing of 720 ft. per minute, the following coefficients of friction:

OIL	PRESSURE, LB. PER SQ. IN.			
	8	16	32	48
Sperm and lard	.159–.25	.138–.192	.086–.141	.077–.144
Olive and cotton seed	.160–.283	.107–.245	.101–.168	.079–.131
Mineral oils	.154–.261	.145–.233	.086–.178	.094–.222

The results show that the coefficient decreases with the pressure within the range reported, but that the results are considerably higher than those for well-lubricated bearings. He also found in connection with the same tests that with *continuous lubrication* sperm oil gave the following coefficients:

PRESSURE, LB. PER SQ. IN.	COEFFICIENT OF FRICTION.
50	.0034
200	.0051
300	.0057

The results of tests of the friction of well-lubricated bearings are summarized by Goodman (*Engineering News*, April 7 and 14, 1888) as follows:

(*a*) *The coefficient of friction of well-lubricated surfaces is from* $\frac{1}{6}$ *to* $\frac{1}{10}$ *that of dry or poorly lubricated surfaces.*

(*b*) *The coefficient of friction for moderate pressures and speeds varies approximately inversely as the normal pressure; the frictional resistance varies as the area in contact, the normal pressure remaining the same.*

(*c*) *For low speeds the coefficient of friction is abnormally high, but as the speed of rubbing increases from about* 10 *to* 100 *ft. per minute, the coefficient of friction diminishes, and again rises when that speed is exceeded, varying approximately as the square root of the speed.*

(*d*) *The coefficient of friction varies approximately inversely as the temperature, within certain limits; namely, just before abrasion takes place.*

149. Method of Testing Lubricants.—To make the matter of the tests of the friction of lubricants clear, it will be convenient to make use of the description of a testing machine used by Dean W. F. M. Goss at Purdue University on graphite, and a mixture of graphite and sperm oil. In making the tests the apparatus shown in Figs. 171 and 172 was used. (See "A Study in Graphite," Joseph Dixon Crucible Co.)

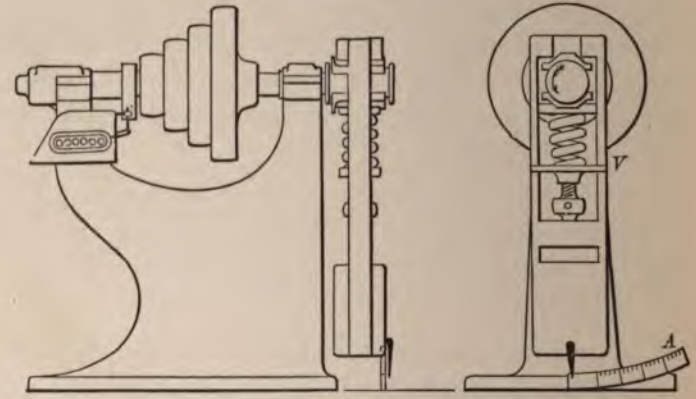


FIG. 171

This apparatus represents, in principle, the machines generally used for testing lubricants. It is therefore shown in some detail. The weight G is hung from the shaft upon which it is suspended by the form of box to be tested. The desired speed of rubbing is obtained by means of the cone of pulleys, and the pressure on the bearing is adjusted by the spring. The temperature of the bearing is read from the thermometer inserted in the bearing. When rotation takes place, the weight G is rotated a certain distance dependent upon the friction. This distance is measured on the scale A. The forces acting upon the pendulum G are shown in Fig. 172, where R represents the resistance of the spring, F the force of friction, l the distance of the center of gravity of G from the axis of rotation, ϕ the angle through which G is deflected, and r the radius of the shaft. Taking moments about the center of the shaft, we have, when G is held in the position shown, due to friction,

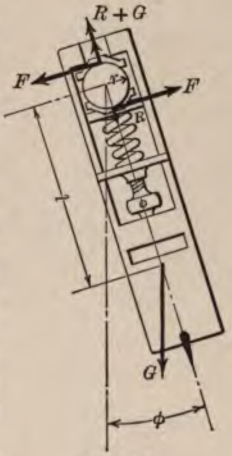


FIG. 172

$F_1 r = Gl \sin \phi$, where F_1 is the total friction on the bearing $2F = F_1$;

but
$$f = \frac{F_1}{R},$$

so that
$$f = \frac{Gl \sin \phi}{rR}.$$

It is customary to take G small compared with R, so that the pressure on both sides of the bearing may be considered equal to R, the resistance of the spring. The spring is easily calibrated so that R may be made anything desired by compressing the spring through the appropriate distance as indicated on the scale V (Fig. 171). The quantities G, l, r, and R are known, and ϕ can be read so that f can be calculated.

If G is not small compared to R, then

$$f=\frac{F_1}{\text{average pressure}}=\frac{2F_1}{(R+G)+R}=\frac{F_1}{R+\frac{G}{2}},$$

so that
$$f=\frac{Gl\sin\phi}{r\left(R+\frac{G}{2}\right)R}.$$

The results of tests made upon a mixture of graphite and oil as a lubricant are given in the pamphlet. The tests were run under 200 lb. per square inch pressure, at a speed of rubbing of 145 ft. per minute. Oil was dropped into the bearing at the rate of about 12 drops per minute, showing a coefficient of friction of $\frac{1}{4}$.

Problem 234. If the weight of the pendulum is 360 lb., the diameter of the shaft $4\frac{1}{2}$ in., distance of the center of gravity of G from the center of shaft 2 ft., the angle ϕ 5 degrees, and the average resistance of the spring 1000 lb., find the coefficient of friction. The weight G should not be neglected in this case.

150. Rolling Friction.—The resistance offered to the rolling of one body over another is known as rolling friction. It is entirely different from sliding friction, and its laws are not so well understood. When a wheel or cylinder (Fig. 173) rolls over a track the track is depressed and the wheel distorted. The force P necessary to overcome this depression and distortion is known as *rolling friction*.

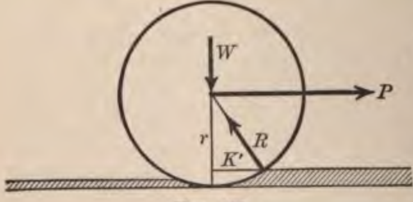


FIG. 173

The forces acting on the wheel are seen from Fig. 173 to be: P the working force, W the weight on the wheel, and R the resistance of the track or roadway to the rolling. This upward pressure R is not quite vertical, but has its point of application a short distance K' from the vertical. Its line of action passes through the center of the wheel. The distance K' depends chiefly upon the roadway; it is called the *coefficient of rolling friction.* It is measured in inches and is not a coefficient of friction in the strict sense that f is the coefficient of sliding friction. Taking moments about the point of application of R, we have, approximately,

$$WK' = Pr,$$

so that

$$K' = \frac{Pr}{W}, \text{ or } P = \frac{KW}{r}.$$

When the track or roadway is elastic or nearly so, we have a condition something like that represented in Fig. 174. The wheel sinks into the material and pushes it ahead, at the same time it comes up behind the wheel. For a portion of the wheel on each side of the point O the roadway is simply compressed; over the remainder of the surface in contact, however, slipping occurs, as indicated by the arrows. The resultant resistance, however, is in front of the vertical through the center,

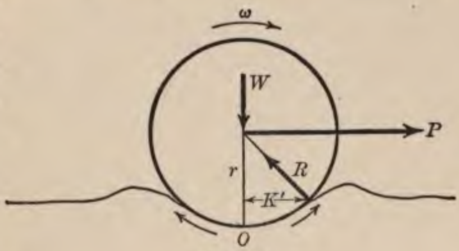


Fig. 174

and we have, as in the case of imperfectly elastic roadways,

$$P = \frac{K' W}{r}.$$

It has been found by Reynolds (see Phil. Trans. Royal Soc., Vol. 166, Part 1) that when a cast-iron roller rolls on a rubber track, the slippage, due to the elasticity of the track, may amount to as much as .84 in. in 34 in. An elastic roller rolling on a hard track will roll less than the geometrical distance traveled by a point on the circumference. When the roller and tracks are of the same material, the roller rolls through less than its geometrical distance.

151. Friction Wheels. — The friction of bearings is often made much less by the use of friction wheels. The arrangement is usually something like that shown in Fig.

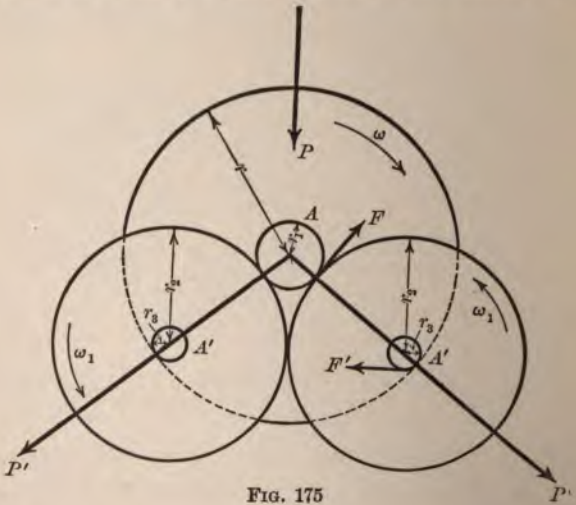


FIG. 175

175. The two friction wheels A' carry the shaft of the mechanism A. The friction of the shaft A is thus changed from sliding to rolling friction. Let P be the normal pressure on the shaft, and let the two equal forces P' acting through the centers of the friction wheels be the components of P acting on the friction wheels. The forces acting on each friction wheel are, then, the pressure P', the friction F, and the friction of the bearing F'. Since P, P', and P' form a balanced system of forces, when the forces acting on the shaft are considered,

$$P' = \frac{P}{2\cos\beta},$$

where 2β is the angle between the forces P'. The value of the friction F' of the friction wheel bearings is

$$F' = f(P' + P') = \frac{fP}{\cos\beta},$$

where f is the coefficient of sliding friction, and the moment of this friction is

$$F'r_3 = \frac{fPr_3}{\cos\beta}.$$

Taking moments about the center of a friction wheel, we have

$$Fr_2 = F'r_3,$$

so that

$$F = \left(\frac{r_3}{r_2}\right)\frac{fP}{\cos\beta}.$$

It is seen that if the ratio $\frac{r_3}{r_2}$ is constant, the friction may be made less by taking β small, so that $\cos\beta$ is large. If r_2 is large as compared with r_3, the friction is reduced.

The work lost due to friction per revolution of A is $2\,\pi r_1 F$, or

$$W = 2\,\pi r_1 \left(\frac{r_3}{r_2}\right) \frac{fP}{\cos\beta}.$$

The friction of A when resting in an ordinary bearing would be fP. In order that the friction of the friction wheels may be less than that of a plain bearing, we must have

$$\frac{r_3}{r_2 \cos\beta} < 1, \text{ or } \frac{r_3}{r_2} < \cos\beta.$$

The work lost, per revolution of A, in a plain bearing, would be $2\,\pi r_1 fP$. It is seen that the criterion that the work lost in the friction bearing be less than that lost in the plain bearing is the same as that given above, viz.,

$$\cos\beta > \frac{r_3}{r_2}.$$

If the angle β is zero, that is, if there is only one friction wheel, so that the center of A is vertically over A', the friction is

$$F = \frac{r_3}{r_2} fP.$$

This is always less than the friction of a plain bearing, since $\frac{r_3}{r_2}$ is always less than unity.

Problem 235. If $P = 4$ tons and the radius of the shaft is 2 in. and the coefficient of friction is .07, what work is lost per revolution? If the shaft makes 3 revolutions per second, what horse power per revolution is lost in friction? Given also $\beta = 45°$, $r_3 = \frac{3}{4}$ in., and $r_2 = 4$ in.

Problem 236. In the case of the shaft mentioned in the preceding problem, how much more horse power per revolution would it take if the bearing was plain? What value of β would give the same loss due to friction in both the plain bearing and the one provided with friction wheels?

152. Resistance of Ordinary Roads. — Resistance to traction consists of axle friction, rolling friction, and grade resistance. Axle friction varies from .012 to .02 of the load, for good lubrication, according to Baker. The tractive power necessary to overcome axle friction for ordinary American carriages has been found to be from 3 lb. to $3\frac{1}{2}$ lb. per ton, and for wagons with medium-sized wheels and axles from $3\frac{1}{2}$ lb. to $4\frac{1}{2}$ lb. per ton.

The total tractive force per ton of load, for wheels 50 in., 30 in., and 26 in., in diameter, respectively, is, according to Baker (*Engineering News*, March 6, 1902):

	Tractive Force in Pounds		
On macadam roads	57	61	70
On timothy and blue grass sod, dry, grass cut .	132	145	179
On timothy and blue grass sod, wet and springy .	173	203	288
On plowed ground, not harrowed, dry and cloddy	252	303	374

Rolling resistance is influenced by the width of the tire. According to Baker, poor macadam, poor gravel, compressible earth roads, and, on agricultural lands, narrow tires, usually give less traction. On earth roads composed of dry loam with 2 to 3 in. of loose dust, traction with $1\frac{1}{2}$-in. tires was 90 lb. per ton, and with 6-in. tires 106 lb. per ton. On the same road when it was hard and dry, with no dust, that is, when it was compressible, the traction was found to be 149 lb. per ton with $1\frac{1}{2}$-in. tires and 109 lb. per ton with 6-in. tires. On broken stone roads, hard and smooth, with no dust or loose stones, the traction per ton was 121 lb. with $1\frac{1}{2}$-in. tires, and 98 lb. with 6-in. tires. Moisture on the surface or mud increases the traction.

Morin found that with 44-in. front and 54-in. rear wheels on hard dry roads the traction per ton was 114 lb. with either $1\frac{3}{4}$-in. or 3-in. tires. On wood-block pavements the traction per ton was 28 lb. with $1\frac{1}{2}$-in. tires, and 38 lb. with 6-in. tires.

On asphalt, bricks, granite, macadam, and steel-road surfaces, investigated by Baker, the traction per ton varied from 17 lb. to 70 lb., the average being 38 lb.

Morin gives the coefficient of rolling friction for wagons on *soft soil* as .065 in., and on *hard roads* .02 in. According to Kent ("Pocket-Book"), tests made upon a loaded omnibus gave the following results:

PAVEMENT	SPEED, MILES PER HOUR	COEFFICIENT, INCHES	RESISTANCE, PER TON, IN LB.
Granite	2.87	.007	17.41
Asphalt . . . . . , .	3.56	.0121	27.14
Wood	3.34	.0185	41.60
Macadam, graveled . .	3.45	.0199	44.48
Macadam, granite, new .	3.51	.0451	101.09

Problem 237. Compare the resistance offered to a load of two tons pulled over asphalt, macadam, good earth roads, or wood-block pavement. Width of tires, 6 in.

Problem 238. Compare the resistances in the above problem with that of steel rails to the same load.

153. Roller Bearings. — In the roller bearings the shaft rolls on hardened steel rollers as shown in cross section in Fig. 176. The rollers are kept in place in some way similar to that shown in the journal of Fig. 177. Such bearings are used where heavy loads are to be carried. Tests of roller bearings have been made by Dean C. H. Benjamin (*Machinery*, October, 1905), who determined the following values for the coefficient of friction. Speed 480 revolutions per minute.

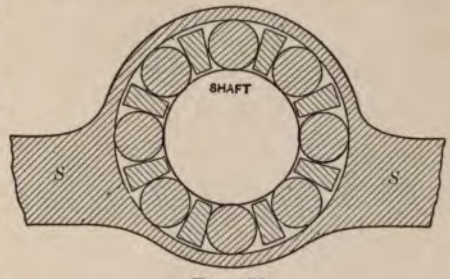


FIG. 176

DIAMETER OF JOURNAL, IN INCHES	ROLLER BEARING			PLAIN CAST-IRON BEARING		
	Max.	Min.	Average	Max.	Min.	Average
$1\frac{15}{16}$	.036	.019	.026	.160	.099	.117
$2\frac{3}{16}$	.052	.034	.040	.129	.071	.094
$2\frac{7}{16}$	.041	.025	.030	.143	.076	.104
$2\frac{15}{16}$	.053	.049	.051	.138	.091	.104

It was found that the coefficient of friction of roller bearings is from $\frac{1}{5}$ to $\frac{1}{3}$ that of plain bearings at moderate speeds and loads. As the load on the bearing increased, the coefficient of friction decreased. Tightening the bearing was found to increase the friction considerably.

Tests of the friction of steel rollers 1, 2, 3, and 4 in. in diameter are reported in the Transactions Am. Soc. C. E., August, 1894. The rollers were tested between plates $1\frac{1}{2}$ in. thick and 5 in. wide, arranged as shown in Fig. 178. Tests were made with the plates and rollers of cast iron, wrought iron, and steel. The friction P' for unit load P was found to be $\frac{.0063}{\sqrt{r}}$ for cast-iron rollers and plates, $\frac{.0120}{\sqrt{r}}$ for wrought iron, and $\frac{.0073}{\sqrt{r}}$ for steel, when r represents the radius of the roller in inches. The rollers were turned and the plates planed, but neither were polished.

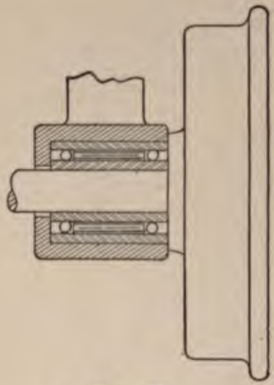


FIG. 177

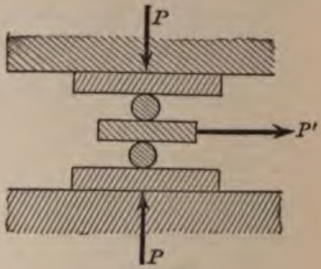


FIG. 178

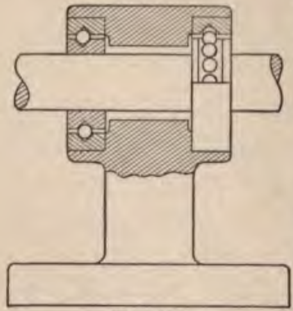


FIG. 179

154. Ball Bearings. — For high speeds and light or moderate loads the friction is much reduced by the use of hardened steel balls instead of the steel rollers. These bearings are now used on all classes of machinery, giving a much greater efficiency except for heavy loads. The principal objection to the ball bearing seems to be due to the fact that there is so little area of contact between the balls and bearing plates. This gives rise to very high stresses over these areas, and consequently a considerable deformation of the balls. When the ball has been changed from its spherical form it is no longer free to roll, and the friction increases rapidly. Some authorities consider a load of from 50 to 150 lbs. sufficient for balls varying in size from $\frac{1}{4}$ to $\frac{1}{2}$ inch in diameter. Figure 179 illustrates a type of bearing used for shafts, and Fig. 180 a type used for thrust blocks.

The conclusions reached by Goodman from a series of tests on bicycle bearings (Proc. Inst. C. E., Vol. 89) are as follows:

(1) *The coefficient of friction of ball bearings is constant for varying loads, hence the frictional resistance varies directly as the load.*

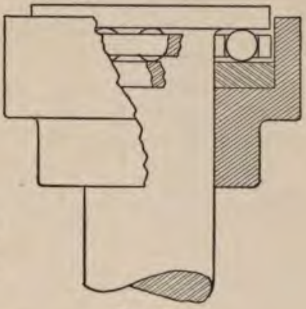


FIG. 180

(2) *The friction is unaffected by a change of temperature.*

The bearings were oiled before starting the tests. The coefficient of friction for ball bearings was found to be rather higher than for plain bearings with bath lubrication, but lower than for ordinary lubrication. Ball bearings will also run easily with a less supply of oil. The following table gives the results of tests of ball bearings. The bearings were oiled before starting, and the tests were run at a temperature of 68° F.

Load on Bearing in Lb.	19 Revolutions per Min.		157 Revolutions per Min.		350 Revolutions per Min.	
	Coeff. friction	Friction, lb.	Coeff. friction	Friction, lb.	Coeff. friction	Friction, lb.
10	.0060	.06	.0105	.10	.0105	.10
20	.0045	.09	.0067	.13	.0120	.24
30	.0050	.15	.0050	.15	.0110	.33
40	.0052	.21	.0052	.21	.0097	.39
50	.0054	.27	.0054	.27	.0090	.45
60	.0050	.30	.0055	.33	.0075	.45
70	.0049	.34	.0054	.38	.0068	.47
80	.0048	.38	.0062	.49	.0060	.48
90	.0050	.45	.0068	.61	.0060	.54
100	.0058	.58	.0069	.69	.0057	.57
110	.0054	.59	.0065	.71	.0060	.66
120	.0055	.66	.0075	.90	.0057	.68
130	.0058	.75	.0078	1.01	.0062	.81
140	.0056	.78	.0077	1.08	.0060	.84
150	.0060	.90	.0083	1.24	.0062	.93
160	.0075	1.20	.0081	1.29	.0058	.93
170	.0079	1.34	.0078	1.33	.0055	.93
180	.0079	1.42	.0078	1.40	.0053	.95
190	.0087	1.65	.0076	1.44	.0054	1.03
200	.0090	1.80	.0081	1.62	.0060	1.20

Another series of tests, run with a constant load on the bearing of 200 lb. and a temperature of 86° F., shows the variation of the coefficient of friction with the speed. It is seen that as the speed *increased* the coefficient and the friction *decreased*. The preceding table, however, shows, *for loads below* 175 *lb., an increase in the coefficient with increase in speed*. In particular, this table shows that for loads below 80 lbs. the coefficient increased with increase of speed; for loads between 90 and 175 lbs. it increased when the speed was 150 R.P.M. and decreased when it was 350 R.P.M. Beyond 175 lbs. the coefficient increased.

Revolutions per Minute	Coefficient Friction	Friction Pounds
15	.00735	1.47
93	.00465	.93
175	.00375	.75
204	.00345	.69
280	.00300	.60

It seems from the data given that the first conclusion of Goodman's should be changed to read: the *coefficient of friction of ball bearings is constant for varying loads, up to a certain limit, beyond which it increases with increase of load.* This limit is about 150 lb. in the tests reported.

Tests on ball bearings designed for machinery subjected to heavy pressures have been made in Germany (see *Zeitschrift des Vereins deutsche Ingenieure*, 1901, p. 73). It was found that at speeds varying from 65 to 780 revolutions per minute, where the bearing was under pressures varying from 2200 lb. to 6600 lb., the coefficient of friction varied little and averaged .0015.

Tests of ball bearings made by Stribeck and reported by Hess (Trans. Am. Soc. M. E., Vol. 28, 1907) give rise to the following conclusions: (*a*) the load that may be put upon a bearing is given by the formula

$$P = \frac{cd^2n}{11.02},$$

where P is the load in pounds on a bearing, consisting of one row of balls, c is a constant dependent upon the material of the balls and supporting surfaces and determined experimentally, d the diameter of the balls, the unit being $\frac{1}{8}$ of an inch, and n the number of balls. For modern materials c varies from 5 to 7.5. (*b*) The coefficient of friction varied from .0011 to .0095. It was independent of speed, "within wide limits," and approximated .0015; this was increased to .003 when the load was about one tenth the maximum.

The following values for the coefficient of friction for heavy loads are reported, from observation, with the statement that the real values are probably somewhat less:

Revolutions per minute	65	100	190	380	580	780	1150
Coefficient of friction for load 840 lb.	.0095	.0095	.0093	.0088	.0085		.0074
Coefficient of friction for load 2400 lb.	.0065	.0062	.0058	.0053	.0050	.0049	.0047
Coefficient of friction for load 4000 to 9250 lb.	.0055	.0054	.0050	.0050	.0041	.0041	.0040

It should be remembered that the friction of a ball bearing is due to both sliding and rolling friction, the sliding friction being due to the elasticity of the balls and the bearing (see Art. 150). Rolling friction is most nearly approached when the balls are hard and not easily changed from their spherical shape. All materials, however, are deformed under pressure so that perfect rolling friction is impossible. On account of the sliding friction present in roller and ball bearings, it is necessary to use a lubricant to prevent wear.

Problem 239. How many ¾-in. balls will be necessary in a ball bearing designed to carry 4000 lb., if $c = 7.5$? If $f = .0015$, what work is lost per revolution, the distance from the axis of rotation to the center of balls being one inch?

155. Friction Gears. — In the friction gears the driver is usually the smaller wheel, and when there is any difference in the materials of which the wheels are made, the driver is made of the softer material. This latter arrangement is resorted to, to prevent flat places being worn on either wheel in case of slipping. These gears have been used for transmitting light loads at high speeds, where toothed gears would be very noisy, or in cases where it is necessary to change the speed or direction of the motion quickly.

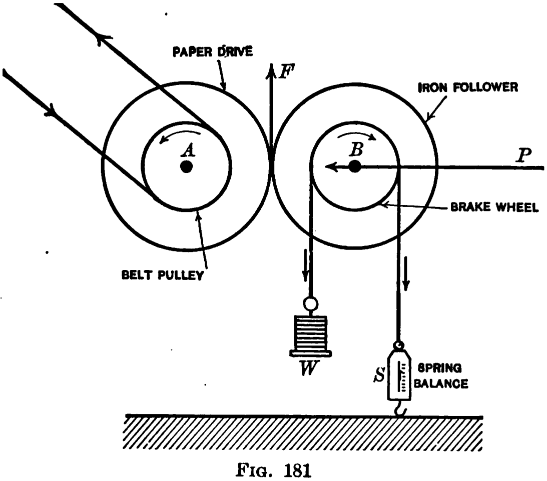


Fig. 181

The use of paper drivers has made possible the transmission of much heavier loads by means of such gears.

A series of tests, made by W. F. M. Goss, and reported in Trans. Am. Soc. M. E., Vol. 18, on the friction between paper drivers and cast-iron followers, is of interest in this connection. The apparatus used is shown in Fig. 181. The pressure between the wheels was obtained by a mechanism that forced the two wheels together with a pressure P. A brake wheel shown in the figure absorbed the power transmitted.

The coefficient of friction was regarded as the ratio of F to P, as in sliding friction. While this is customary, it is not entirely true, since we have the rolling of one body over the other. We shall, however, assume that we may call the coefficient of friction $f=\frac{F}{P}$. It was found that the coefficient of friction varied with the slippage, but was fairly constant for all pressures up to some point between 150 to 200 lb. per inch of width of wheel face. "*Variations in the peripheral speed between* 400 *and* 2800 *ft. per minute do not affect the coefficient of friction.*"

If the allowable coefficient of friction be taken as .20, the horse power transmitted per inch of width of face of the wheel is

$$\text{H.P.} = \frac{150 \times .2 \times \frac{1}{12}\pi d \times w \times N}{33{,}000} = .000238\, dwN,$$

where d is the diameter of the friction wheel in inches, w the width of its face in inches, and N the revolutions per minute. Using this formula, the following table is given in the article in question:

HORSE POWER WHICH MAY BE TRANSMITTED BY MEANS OF PAPER FRICTION WHEEL OF ONE INCH FACE, WHEN RUN UNDER A PRESSURE OF 150 LB.

DIAMETER OF PULLEY IN INCHES	REVOLUTIONS PER MINUTE							
	25	50	75	100	150	200	600	1000
8	.0476	.0952	.1428	.1904	.2856	.3808	1.1424	1.904
10	.0595	.1190	.1785	.2380	.3570	.4760	1.4280	2.380
14	.0833	.1666	.2499	.3332	.4998	.6664	1.9992	3.332
16	.0952	.1904	.2856	.3808	.5712	.7616	2.2848	3.808
18	.1071	.2142	.3213	.4284	.6426	.8568	2.5704	4.288
24	.1428	.2856	.4284	.5712	.8568	1.1424	3.4272	5.712
30	.1785	.3570	.5355	.7140	1.0710	1.4280	4.2840	7.140
36	.2142	.4284	.6426	.8568	1.2852	1.7136	5.1408	8.560
42	.2499	.4998	.7497	.9996	1.4994	1.9992	5.9976	9.996
48	.2856	.5712	.8568	1.1424	1.7136	2.2848	6.8544	11.420

The value of the coefficient of friction for friction gears, (Kent, "Pocket Book") may be taken from .15 to .20 for metal on metal; .25 to .30 for wood on metal; .20 for wood on compressed paper.

Problem 240. If the friction wheels are grooved as shown in Fig. 182, both of cast iron, and the small driver fits into the groove of the larger follower, we may take $f = .18$. Then

$$F = 2fN = 2fP \cos \alpha$$
$$= .36\ P \cos \alpha.$$

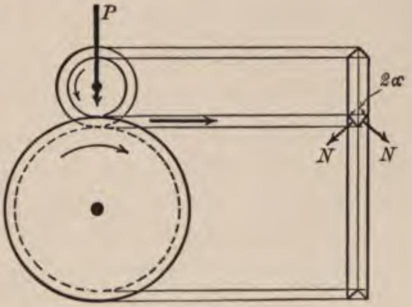


FIG. 182

Problem 241. The speed of the rim of two grooved friction wheels is 400 ft. per minute. If $\alpha = 45°$, $f = .18$, what must be the pressure P to transmit 100 horse power?

Problem 242. What horse power may be transmitted by the gearing in the preceding problem, if $P = 6000$ lb. and the peripheral velocity is 12 ft. per second?

156. Friction of Belts. — When a belt or cord passes over a pulley and is acted upon by tensions T_1 and T_2, the tensions are unequal, due to the friction of the pulley on the belt. We shall determine the relation between T_1 and T_2. Let the pulley be represented in Fig. 183. The belt covers an arc of the pulley whose angle is α. Consider the forces acting upon the belt and suppose T_1 and T_2 to be the tensions in the belt on the tight and slack sides, respectively, and T the tension in the belt at any point of the arc of contact. Let F be the total friction between the pulley and belt and dF the friction on an elementary arc ds. If dP is the normal pressure on an elementary arc, then $dF = fdP$ and $T_1 - T_2 = F$, where f is the coefficient of friction.

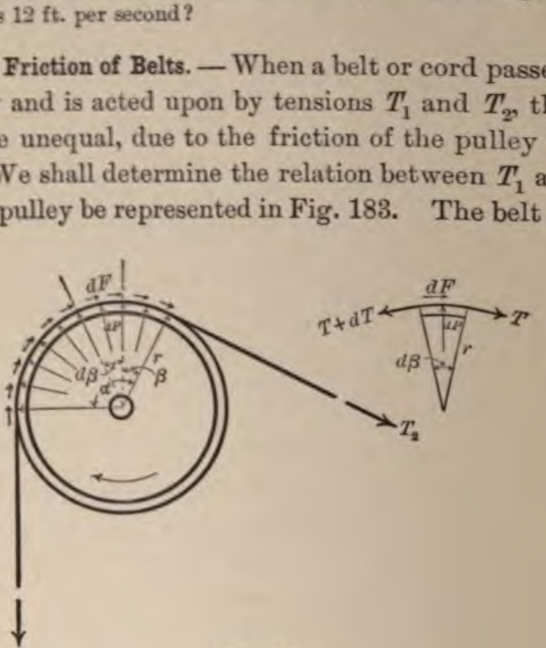


FIG. 183

Represent as in Fig. 183 an elementary arc of the belt of length ds. The forces acting on this elementary part are, the tensions $T+dT$ and T, the friction dF, and the normal pressure dP. Taking moments about the center of the pulley, we have

$$(T+dT)r = dFr + Tr,$$

or

$$dT = dF.$$

Of the forces acting upon this elementary portion of the belt, dT and dF are in equilibrium, so that T, dP, and T must also be in equilibrium. Since this is true, these latter forces must form a closed triangle when drawn to scale (Art. 13). We have, then, from Fig. 184, approximately,

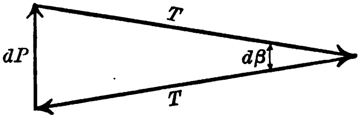


Fig. 184

$$dP = Td\beta,$$

so that

$$dT = dF = fTd\beta,$$

or

$$\int_{T_2}^{T_1} \frac{dT}{T} = \int_0^{\alpha} fd\beta.$$

This gives

$$\log_e T_1 - \log_e T_2 = f\alpha,$$

or

$$\log_e \frac{T_1}{T_2} = f\alpha;$$

or, writing it in the exponential form,

$$T_1 = T_2 e^{f\alpha}.$$

This is the relation desired. The quantity $e = 2.72^+$ is the base of the system of natural logarithms. The $\log_{10} e = .4343$.

$$F = T_1 - T_2 = T_1(1 - e^{-f\alpha}) = T_2(e^{f\alpha} - 1).$$

When the band is used to resist the motion of a pulley as in some types of brakes (see Fig. 185), it is known as a friction strap, see Art. 165.

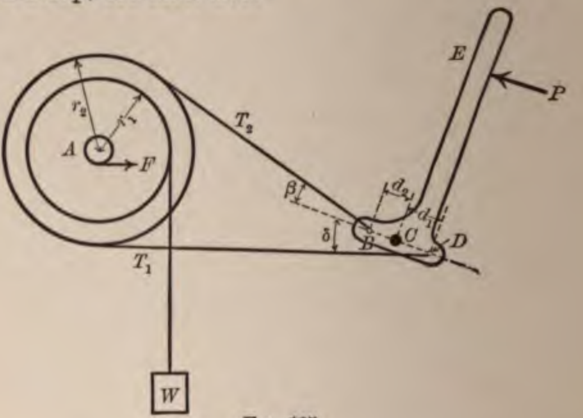


FIG. 185

Problem 243. A rope is wrapped four times around a post and a man exerts a pull of 50 lb. on one end. If the coefficient of friction is .3, what force can be exerted upon a boat attached to the other end of the rope?

Problem 244. A pulley 4 ft. in diameter, making 200 revolutions per minute, drives a belt that absorbs 20 horse power. What must be the width of the belt in order that the tension may not exceed 70 lb. per inch of width?

Problem 245. What should be the width of a belt $\frac{1}{4}$ of an inch thick to transmit 10 horse power? The belt covers .3 the smaller pulley and has a velocity of 500 ft. per minute. The coefficient of friction is .27 and the strength of the material 300 lb. per square inch.

NOTE. The power that can be transmitted by a belt depends upon the friction between the belt and pulley. So that

$$\text{H.P.} = \frac{Fv}{33{,}000} = \frac{(T_1 - T_2)v}{33{,}000}.$$

157. Transmission Dynamometer. — It has been shown, in Art. 156, that the tension of a belt on the tight side is greater than the tension on the slack side. The transmission dynamometer (the Fronde dynamometer), illustrated in Fig. 186, is designed to measure the difference in these tensions. Let the pulley *D* be the driver and the pulley *E* the follower, so that T_1 represents the tight side of the belt and T_2 the slack side. The pulleys *B*, *B* run loose on the T-shaped frame *CBB*. This frame is pivoted at *A*. If we neglect the friction due to the loose pulleys, we have the following forces acting on the T-frame, two forces T_1 at the center of the right-hand pulley *B*, two forces T_2 at the center of the left-hand pulley *B*, a measurable reaction *P* at *C*, and the reaction of the pin at *A*. Taking moments about the pin, we have

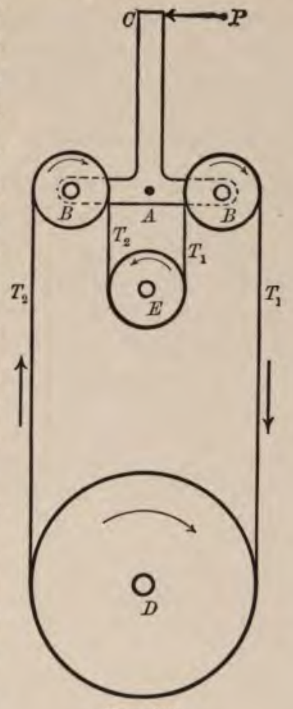


FIG. 186

$$P(CA) = 2\,T_1(BA) - 2\,T_2(BA)$$
$$= 2\,BA(T_1 - T_2),$$

so that
$$T_1 - T_2 = \frac{P}{2}\frac{CA}{BA}.$$

The distances *CA* and *BA* are known, and *P* may be measured; the difference, then, $T_1 - T_2$, may always be obtained. The value $T_1 - T_2$ is then known and the horse power determined by the relation (see Problem 253)

$$\text{H.P.} = \frac{(T_1 - T_2)\, 2\pi rn}{33{,}000},$$

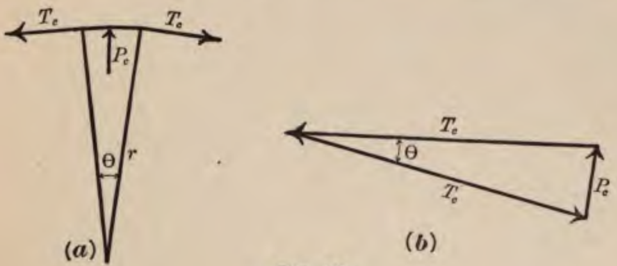
where n is the number of revolutions, r is the radius of the machine pulley in feet.

158. Creeping or Slip of Belts. — A belt that transmits power between two pulleys is tighter on the *driving side* than it is on the *following side*. On account of this difference in tension and the elasticity of the material, the tight side is stretched more than the slack side. To compensate for this greater stretch on one side than on the other, the belt *creeps* or *slips* over the pulleys. This slip has been found for ordinary conditions to vary from 3 to 12 ft. per minute. The coefficient of friction when the slip is considered is about .27 (Lanza). It has also been found that the loss in horse power in well-designed belt drives, due to slip, does not exceed 3 or 4 per cent of the gross power transmitted, and that ropes are practically as efficient as belts in this respect. For an account of the experimental investigations on this subject the student is referred to Institution of Mechanical Engineers, 1895, Vols. 3–4, p. 599, and Transactions Am. Soc. M. E., Vol. 26, 1905, p. 584.

159. Coefficient of Friction of Belting. — The value of the coefficient of friction of belting depends, not only on the slip but also upon the condition and material of the rubbing surfaces. Morin found for leather belts on iron pulleys the coefficient of friction $f = .56$ when dry, .36

when wet, .23 when greasy, and .15 when oily (Kent, "Hand Book"). Most investigators, however, including Morin, took no account of slip, so that the best value of f, everything considered, is that given in the preceding article (.27).

160. Centrifugal Tension of Belts. — When a belt runs at a high rate of speed over a pulley there is considerable tension introduced in the belt due to the centrifugal force. We have seen (Art. 86) that the centrifugal force equals $\frac{Mv^2}{r}$, where M is the mass and v the tangential velocity. Let the centrifugal force be represented by P_c and the tension in the belt due to this force by T_c. We know that $P_c = \frac{Mv^2}{r}$. Now if we consider a section of belt one foot long and of one square inch cross section, we may consider the tensions T_c, at either end of this length, in equilibrium with P_c (see Fig. 187 (a)). From Fig. 187 (b) we have approximately $P_c = T_c\theta$, but from Fig. 187 (a), $\theta = \frac{1}{r}$, so that $P_c = \frac{T_c}{r.}$ Since $P_c = \frac{Mv^2}{r}$,



(a) (b)

FIG. 187

$$T_c = Mv^2 = \frac{Wv^2}{g},$$

where W is the weight of a portion one foot long and one square inch in cross section. If γ for leather is 56 lb.,

$$W = .388 \text{ lb. and } T_c = \frac{.388}{32.2}v^2 = .012\ v^2.$$

Hence, in designing belts, the total tension must be

$$T_1 + T_c = T_1 + .012\ v^2 = T_2 e^{fa} + .012\ v^2.$$

Problem 246. A belt runs at a velocity of 4000 ft. per minute. What tension is introduced by the centrifugal force of the belt in passing over the wheel?

Problem 247. What additional width of belt must be provided for in Problems 244 and 245 if the centrifugal force of the belt is considered?

161. Stiffness of Belts and Ropes. — Belts and ropes used in the transmission of power are not perfectly flexible, so that some force is necessary to bend them around the pulleys. We desire to know the magnitude of this force. Let T (Fig. 188) be the tension in the on-side of the belt and $T + T_1$ the tension on the off-side. Neglecting the effect of the friction of the pulley, T_1 represents the force necessary to overcome the stiffness of the rope. In the analysis here given, it is assumed that while

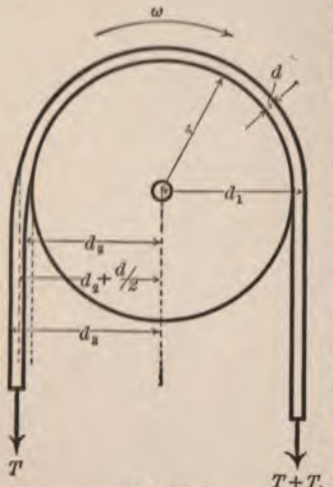


Fig. 188

it takes a certain force T_1 to cause the rope to wind around the pulley, it does not require any force to straighten it. This assumption is nearly true for steel wire rope, but not nearly so true for hemp rope. All rope requires some force to straighten it when coming off the pulley.

Taking moments about the center of the pulley and neglecting the friction of the bearing, we have

$$(T+T_1)d_1 = Td_3,$$

or

$$T_1 = T\frac{(d_3-d_1)}{d_1},$$

where

$$d_1 = r+\frac{d}{2}, \text{ and } d_3 = r+a_1+\frac{d}{2}+a_2.$$

The distance a_1 is due to the stiffness of the rope, and the distance a_2 the distance of the point of application of T from the center of the rope. That T does not act at the center of the rope, but at a distance a_2 toward the outside, is due to the fact that the outside of the rope is under greater tension than the inside. Now the distance a_1 for inelastic ropes decreases as T increases, and so we may write $a_1 = \frac{c_1}{T}$, where c_1 is a constant, determined experimentally. For wire rope, a_1 increases with increased radius of the pulley, and decreases with increased tension, so that we may write

$$a_1 = \frac{c_1\left(r+\frac{d}{2}\right)}{T};$$

making these substitutions in the above equation, we have

$$T_1 = \frac{c_1+a^2T}{r+\frac{d}{2}} \text{ for hemp rope,}$$

and

$$T_1 = e_1 + \frac{a_2 T}{r + \frac{d}{2}} \text{ for wire rope.}$$

For *tarred* hemp ropes, e_1 has been found (see Du Bois, "Mechanics of Engineering") to be 100, and a_2, .222, so that

$$T_1 = \frac{100 + .222\,T}{r + \frac{d}{2}} \text{ pounds.}$$

For *new hemp ropes*,

$$T_1 = \frac{4 + .0645\,T}{r + \frac{d}{2}} \text{ pounds.}$$

For *wire ropes*,

$$T = 1.08 + \frac{.0937\,T}{r + \frac{d}{2}} \text{ pounds.}$$

In each case T is expressed in pounds and r and d in inches.

Problem 248. — A new hemp rope, one inch in diameter, passes over a pulley 13 in. in diameter, under a tension of 500 lb. What is the force necessary to overcome the stiffness of the rope? What per cent is this of the total tension in the rope?

Problem 249. — A wire rope, one inch in diameter, passes over a pulley 2 ft. in diameter, under a tension of 1000 lb. What force is necessary to overcome the stiffness of the rope? Compare this force with that necessary to overcome the stiffness of the same rope under the same tension, when it passes over a pulley 12 in. in diameter. What per cent of the total tension is it in each case?

Problem 250. — A new hemp rope, 2½ in. in diameter, passes over a grooved pulley 34 in. in diameter, under a tension of 1000 lb. What force is necessary to overcome the stiffness of the rope? Allow an increase of 6 per cent for the grooved pulley.

It will be seen from the formula and the problems that the force necessary to overcome the stiffness of ropes is greater for small pulleys than for large ones.

The following empirical formula will be found useful (see *Memoirs et Compte rendu de la Société des Ingénieurs Civile*, December. 1893, p. 558, or Proc. Inst. C. E., Vol. 116, p. 455):

$$T_1 = \frac{.083\, d^2 T^{.7}}{r^{1.2}}$$

for ropes, where d and r are expressed in millimeters and T_1 and T in kilograms. A formula for belting is also given,

$$T_1 = \frac{t}{r^2}\left[wt + 14\,T\right],$$

where w is the width of the belt and t its thickness. T_1 and T are expressed in kilograms and w, t, and r in millimeters.

The student should solve Problems 248 and 249, using the empirical formula given above.

Problem 251. A belt 12 in. wide and $\frac{1}{2}$ in. thick passes over a pulley 18 in. in diameter under a tension of 1000 lb. What force is necessary to overcome the stiffness of the belt?

In using the empirical formula just given it will be necessary to change pounds to kilograms and distances to millimeters.

162. Friction of a Worn Bearing.—The friction of a bearing that fits perfectly is the friction of one surface sliding over another and is given by the equation

$$F = fN,$$

where F is the force of friction, f the coefficient of friction, and N is the total normal pressure on the bearing.

When, however, the bearing is worn, as is shown much exaggerated in Fig. 189, the friction may be somewhat different. When motion begins, the shaft will roll up on the bearing until it reaches a point A where slipping begins. If motion continues, slipping will continue along a line of contact through A. Let P be a force that causes the rotation, R a force tending to resist the rotation, and R_1 the reaction of the bearing on the shaft. There are only three forces acting on the shaft, so that P, R, and R_1 must meet in the point B. The direction of R_1 is accordingly determined. The normal pressure is $N = R_1 \cos\theta$, and the force of friction is

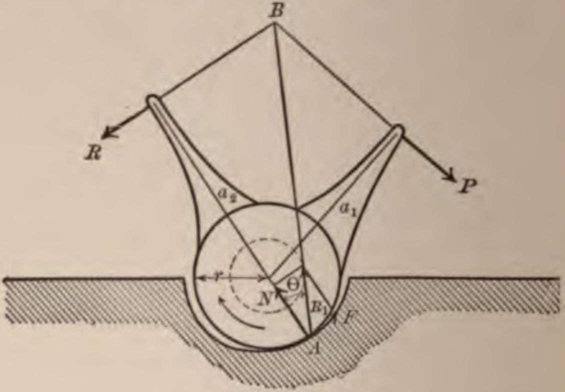


FIG. 189

$$F = R_1 \sin\theta.$$

It is seen that θ is the angle of friction. The moment of the friction with respect to the center of the axle is

$$Fr = R_1 r \sin\theta.$$

If the axle is well lubricated so that θ is small and $\sin\theta$ may be replaced by $\tan\theta = f$, the friction is

$$F = fR_1,$$

and the moment

$$Fr = fR_1 r.$$

The circle tangent to AB radius $r\sin\theta$ is called the *friction circle*. Since r and θ are known generally, this circle may be made use of in locating the point A. In other words, the shaft will continue to rotate in the bearing so long as the reaction R_1 falls within the friction circle, and slipping will begin as soon as the direction of R_1 becomes tangent to the friction circle.

Problem 252. If the radius of the shaft is 1 in., $\theta = 4°$, $P = 500$ lb., $a_1 = 3$ ft., $a_2 = 2$ ft., angle between a_1 and a_2 is 100° and P and R are at right angles to a_1 and a_2, what resistance R may be overcome by P when slipping occurs?

Problem 253. The radius of a shaft is 1 in., $R = 20$ lb., $P = 20$ lb., $a_1 = 3$ ft., and $a_2 = 2$ ft. What force of friction will be acting at the point A, when the angles between P and a_1 and R and a_2 are right angles? What must be the value of the coefficient of friction?

163. Friction of Pivots.—The friction of pivots presents a case of sliding friction, so that the force of friction F equals the coefficient of friction times the normal pressure. That is,

$$F = fN.$$

(*a*) *Flat-End Pivot.* — The friction on a flat-end pivot is greatest on the outside and varies linearly to zero at the center as shown in Fig. 190. The resultant force of friction

R has its point of application $\frac{2}{3}r$ from the center. We may write

$$F = R = fP,$$

and the moment of the friction with respect to the center is

$$\text{Moment} = \tfrac{2}{3}\,rfP.$$

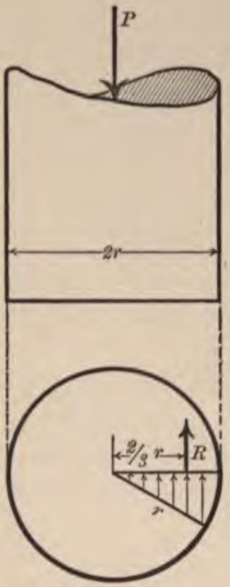


FIG. 190

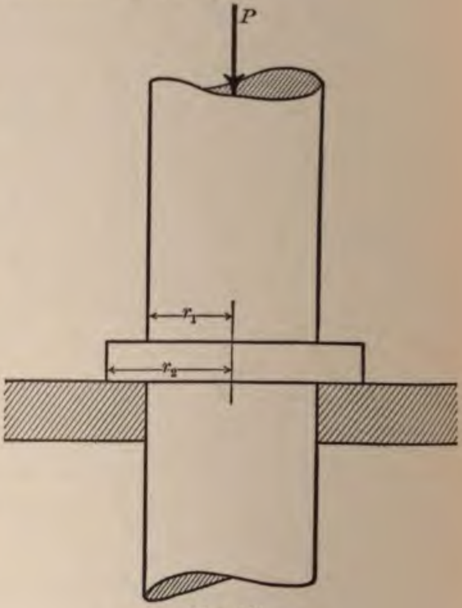


FIG. 191

The work lost per revolution is

$$W = fP(2\,\pi\tfrac{2}{3}\,r) = \tfrac{4}{3}\pi r f P.$$

(*b*) *Collar Bearing or Hollow Pivot.* Let the outside radius be r_2 and the inside radius r_1 (Fig. 191); then,

$$F = R = fP,$$

and the moment of friction is

$$\text{Moment} = \left(\frac{2}{3}\frac{r_2^3 - r_1^3}{r_2^2 - r_1^2}\right) fP.$$

(For the position of the center of gravity, see Art. 24.)

The work lost, due to friction, per revolution is

$$W = \frac{4}{3}\left(\frac{r_2^3 - r_1^3}{r_2^2 - r_1^2}\right)\pi fP.$$

If $r_1 = 0$, this reduces to the work lost per revolution for the solid flat-end pivot.

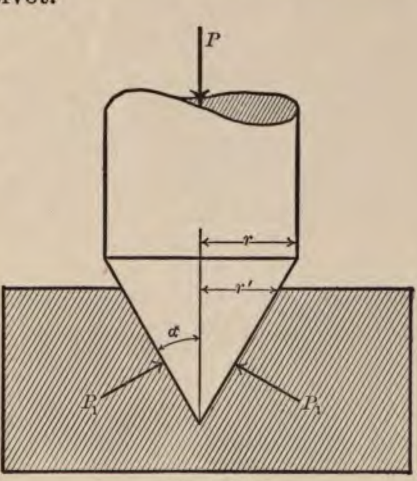


Fig. 192

(*c*) *Conical Pivot.* The conical pivots, such as are illustrated in Fig. 192, do not usually fit into the step the entire depth of the cone. Let the radius of the cone at the top of the step be r', α half the angle of the cone, and P_1 the resultant normal reaction of the bearing on the pivot. Then

$$P_1 = \frac{P}{2 \sin \alpha},$$

and the total friction $F = R = \dfrac{fP}{2 \sin \alpha}$.

The moment of friction in this case is

$$\text{Moment} = \left(\frac{2}{3}r'\right)\left(\frac{fP}{2 \sin \alpha}\right),$$

since the resultant friction may be regarded as applied at $\frac{2}{3}r'$. The work lost due to friction per revolution may be written

$$W=\frac{4}{3}\pi r'\frac{fP}{\sin\alpha}$$

If $\alpha=\frac{\pi}{2}$, this value for work lost reduces to the work lost per revolution, in the case of the flat-end solid pivot. It is easily seen, since $\sin\alpha$ is less than unity, that if r' is nearly equal to r, the friction of the conical bearing is greater than the friction of the flat-end bearing. This might have been expected from the wedgelike action of the pivot on the step. It is also easily seen that r' may be taken small enough so that the friction will be less than the friction of the flat pivot. The work lost due to friction in the case of the conical pivot will be equal to, greater, or less than, the work lost, due to friction in the case of the flat-end pivot, according as

$$r'\overset{=}{\gtrless}r\sin\alpha.$$

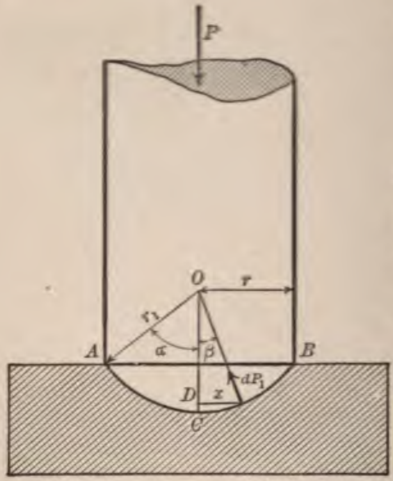


FIG. 193

(*d*) *Spherical Pivot.* Suppose the end of the pivot is a spherical surface, as shown in Fig. 193. Let r be the radius of the shaft and r_1 the radius of the spherical surface; then the load per unit of area of horizontal surface is $\frac{P}{\pi r^2}$.

The horizontal projection of any elementary circle of the bearing, of radius x, is $2\,\pi x dx$. The load on this area is

$$2\,\pi x dx\left(\frac{P}{\pi r^2}\right)=\frac{2\,Pxdx}{r^2},$$

and the corresponding normal pressure is

$$dP_1=\frac{2\,Pxdx}{r^2}\sec\beta.$$

But $\cos\beta=\frac{OD}{r_1}=\frac{\sqrt{r_1^2-x^2}}{r_1}$, so that $dP_1=\frac{2\,Pxdx}{r^2}\left(\frac{r_1}{\sqrt{r_1^2-x^2}}\right)$.

The corresponding friction is fdP_1.

Integrating between the values $x=0$ and $x=r$, the value for the total friction is given by

$$F=R=\int_0^r\frac{2\,Pr_1f}{r^2}\left(\frac{xdx}{\sqrt{r_1^2-x^2}}\right)$$
$$=\frac{2\,Pr_1f}{r_2}(r_1-\sqrt{r_1^2-r^2}).$$

Since $r=r_1\sin\alpha$ and $\sqrt{r_1^2-r^2}=r_1\cos\alpha$, the expression for the friction may be written

$$F=\frac{2fP}{1+\cos\alpha}.$$

If $\alpha=\frac{\pi}{2}$, that is, if the bearing is hemispherical, $F=2fP$, and if $\alpha=0$, that is, if the bearing is flat, $F=fP$.

The moment of the friction is given by adding all the terms fdP_1x by means of integration; this gives

$$\text{Moment}=\frac{2fPr_1}{r^2}\left(\frac{r_1^2}{2}\sin^{-1}\frac{r}{r_1}-\frac{r}{2}\sqrt{r_1^2-r^2}\right),$$

or in terms of α,

$$\text{Moment}=fPr\left(\frac{\alpha}{\sin^2\alpha}-\cot\alpha\right).$$

The work lost due to friction, for each revolution, is found by adding the work lost by friction on each elementary area of the bearing; that is, by finding the sum of such terms as $2\pi x f dP_1$ by means of integration; this gives

$$W = 2\pi f P r \left(\frac{\alpha}{\sin^2 \alpha} - \cot \alpha\right).$$

If the bearing is hemispherical, $\alpha = \frac{\pi}{2}$, and the moment becomes

$$\text{Moment} = \frac{f\pi P r}{2}$$

and the work lost per revolution

$$W = f\pi^2 P r.$$

The friction of flat pivots is often made much less by forcing oil into the bearing, so that the shaft runs on a film of oil. In the case of the turbine shafts of the Niagara Falls Power Company (see Art. 135) the downward pressure is counteracted by an upward water pressure. In some cases the end of a flat pivot has been floated on a mercury bath. This reduces the friction to a minimum (see *Engineering*, July 4, 1893).

The Schiele pivot is a pivot designed to wear uniformly all over its surface. The surface is a tractrix of revolution; that is, the surface formed by revolving a tractrix about its asymptote. Its value as a thrust bearing is not as great as was first anticipated (see *American Machinist*, April 19, 1894).

The coefficient of friction for well-lubricated bearings of flat-end pivots has been found to vary from .0044 to .0221 (see Proc. Inst. M. E., 1891). For poorly lubricated bearings the coefficient may be as high as .10 or .25 for dry bearings.

Problem 254. Show that the work lost per revolution for the hemispherical pivot is 2.35 times the work lost per revolution for the flat pivot.

Problem 255. The entire weight of the shaft and rotating parts of the turbines of the Niagara Falls power plant is 152,000 lb., the diameter of the shaft 11 in. If the coefficient of friction is considered as .02 and the bearing a flat-end pivot, what work would be lost per revolution due to friction?

Problem 256. A vertical shaft carrying 20 tons revolves at a speed of 50 revolutions per minute. The shaft is 8 in. in diameter and the coefficient of friction, considering medium lubrication, is .08. What work is lost per revolution if the pivot is flat? What horse power? What horse power is lost if the pivot is hemispherical?

Problem 257. What horse power would be lost if the shaft in the preceding problem was provided with a collar bearing 18 in. outside diameter instead of a flat-end block? Compare results.

Problem 258. A vertical shaft making 200 revolutions per minute carries a load of 20 tons. The shaft is 6 in. in diameter and is provided with a flat-end bearing, well lubricated. If the coefficient of friction is .004, what horse power is lost due to friction?

164. Absorption Dynamometer.—The friction brake shown in Fig. 181 is used to absorb the energy of the mechanism. It may be used as a means of measuring the energy, and when so used it may be called an absorption dynamometer. The weight W, attached to one end of the friction band, corresponds to the tension in the tight side of an ordinary belt (see Art. 156), while the force measured by the spring S corresponds to the tension on slack side of a belt. Let $W = T_1$ and $S = T_2$; then $T_1 = T_2 e^{fa}$, just as was found in the case of belt tension. The work absorbed per revolution is work $= (T_1 - T_2)2\pi r_1$, where r_1 is the radius of the brake wheel. The horse power absorbed is

$$\text{H.P.} = \frac{(T_1 - T_2)2\pi r_1}{33{,}000}$$

In many cases the friction band is a hemp rope, and in such cases it is possible to wrap the rope one or more times around the pulley, making it possible to make $T_1 - T_2$ large while T_2 is small.

The surface of the brake wheel may be kept cool by allowing water to flow over the inside surface of the rim, which should be provided with inside flanges for that purpose.

165. Friction Brake. — The friction brake shown in Fig. 185 consists of the lever EC, the friction band, and the friction wheel. Such brakes are used on many types of hoisting drums, automobiles, etc. Let the band tensions be T_1 and T_2, and let W be the force causing the motion, that is, the working force, and P the force applied at the end of the lever EC in such a way as to retard the rotation of the drum. We have here as before $T_1 = T_2 e^{fa}$ and the work per revolution $(T_1 - T_2)2\pi r_1 + F2\pi r_3$. By taking moments about A we have $T_1 - T_2 = \dfrac{Wr_2 - Fr_3}{r_1}$, where r_3 is the radius of the shaft and F is the force of friction acting on the shaft. Taking moments about C, we have

$$P_1(EC) = T_1 d_1 \sin\delta + T_2 d_2 \sin\beta.$$

Problem 259. A weight of one ton is being lowered into a mine by means of a friction brake. The radius of the drum is 1½ ft., radius of the friction wheel 2 ft., coefficient of brake friction .30, $\delta = 45°$, $\beta = 15°$, $d_1 = d_2 = 1$ ft., $EC = 6$ ft., radius of shaft 1 in., coefficient of axle friction .04, and the weight of the drum and brake wheel is 600 lb. Find T_1, T_2, and P in order that the weight W may be lowered with uniform velocity.

Problem 260. Suppose the weight in the above problem is being lowered with a velocity of 10 ft. per second when it is discovered that the velocity must be reduced one half while it is being lowered the next 10 ft., what pressure P will it be necessary to apply to the lever at E to make the change? What will be the tension in the friction bands and the tension in the rope that supports W?

Problem 261. If the weight in the above problem has a velocity of 10 ft. per second, and it is required that the mechanism be so constructed that it could be stopped in a distance of 6 ft., what pressure P on the lever and tensions T_1 and T_2 would it require? What would be the tension in the rope caused by the sudden stop? Compare this tension with W, the tension when the motion is uniform.

166. Prony Friction Brake. — The Prony friction brake may be used as an absorption dynamometer as shown in

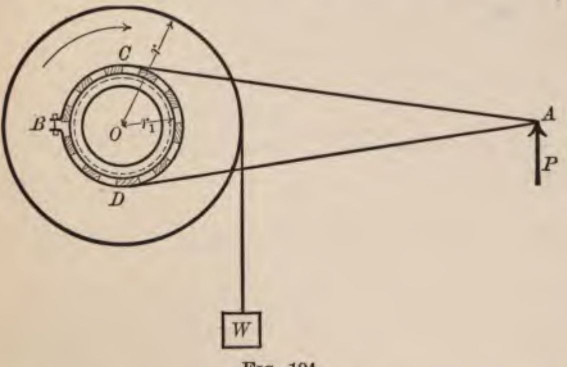


FIG. 194

principle in Fig. 194. Let W be a working force acting on the wheel of radius r and suppose the brake wheel to be of radius r_1. The brake consists of a series of blocks of wood attached to the inner side of a metal band in such a way that it may be tightened around the brake wheel as desired by a screw at *B*. This band is kept from turning by a lever *CAD*, held in the position shown by an upward pressure *P*, at *A*. Considering the forces acting on the brake and taking moments about the center, we have the couple due to friction, Fr_1, equal to the moment $P(OA)$, or

$$Fr_1 = P(OA).$$

Considering the forces acting on the wheel, and neglecting axle friction, we get

$$Fr_1 = Wr.$$

The energy absorbed is used in heating the brake wheel. The wheel is kept cool by water on the inside of the rim. The work absorbed is $2\pi r_1 Fn = 2\pi P(OA)n$, where n is the number of revolutions. The force *P* may be measured by allowing a projection of the arm at *A* to press upon a platform scales. The horse power absorbed is

$$\text{H.P.} = \frac{2\pi P(OA)n}{33{,}000},$$

where OA is expressed in feet, and n is the number of revolutions per minute.

If OA be taken as $\frac{33}{2\pi}$, a convenient length, the formula reduces to

$$\text{H.P.} = \frac{Pn}{1000}.$$

A dynamometer slightly different from the Prony dynamometer is shown in Fig. 195. It differs only in the means of measuring *P*. In this case the force *P* is measured by the angular displacement of a heavy pendulum W_1. Taking moments about the axis of W_1 and calling

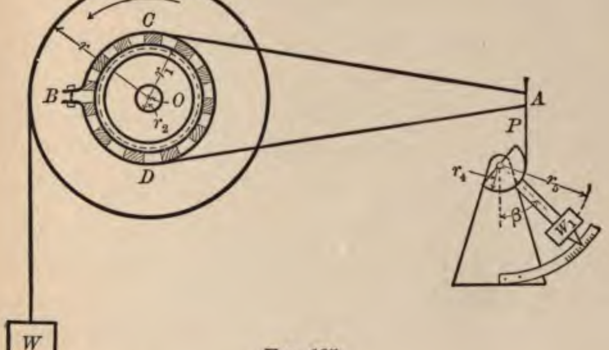


FIG. 195

r_5 the distance from that axis to its center of gravity and β the angular displacement, we have

$$Pr_4 = W_1 r_5 \sin \beta,$$

so that the horse power absorbed may be written

$$\text{H.P.} = \frac{2\pi(OA)W_1 r_5 \sin\beta n}{r_4\, 33{,}000},$$

where OA, r_4, and r_5 are expressed in feet. If OA be taken as $\frac{33}{2\pi}$, this becomes

$$\text{H.P.} = \frac{W_1 r_5 \sin\beta n}{r_4 1000}.$$

The student should understand that the rotation of the mechanism at O is not in every case due to a weight W being acted upon by gravity. In fact, in most cases, the motion will be due to the action of some kind of engine. This, however, will not change the expressions for horse power.

Problem 262. If $W_1 = 100$ lb., $OA = \frac{33}{2\pi}$, $r_5 = 2$ ft., and $r_4 = 6$ in., what horse power is absorbed by the brake if β is 30°, and n is 300 revolutions per minute?

167. Friction of Brake Shoes. — The application of the brake shoe to the wheel of an ordinary railway car is shown in Fig. 196, where F' is the axle friction, F the brake-shoe friction, N the normal pressure of the brake shoe, G the weight on the axle, and F_1 and N_1 the reaction of the rail on the wheel. The brakes on a railway car when applied should be capable of absorbing all the energy of the car in a very short time. The high speeds of modern trains require a system of perfectly working brakes, capable of stopping the car when running at its maximum speed in a very short distance.

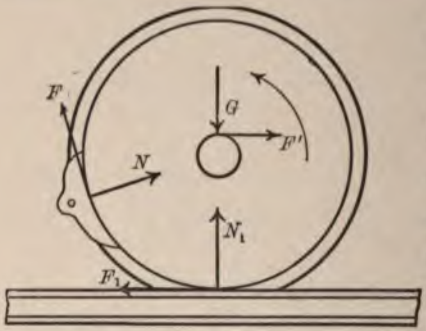


FIG. 196

The coefficient of friction between the shoes and wheel for cast-iron wheels at a speed of 40 mi. per hour is about $\frac{1}{4}$, while at a point 15 ft. from stopping the coefficient of friction is increased 7 per cent, or it is about .27. The coefficient for steel-tired wheels at a speed of 65 mi. per hour is .15, and at a point 15 ft. from stopping it is .10. (See Proc. M. C. B. Assoc., Vol. 39, 1905, p. 431.)

The brake shoes act most efficiently when the force of friction F is as large as it can be made without causing a slipping of the wheel on the rail (skidding). The normal pressure N, corresponding to the values of the coefficient of friction given above, varies in brake-shoe tests from 2800 lb. to 6800 lb., sometimes being as high as 10,000 lb.

Problem 263. A 20-ton car moving on a level track with a velocity of a mile a minute is subjected to a normal brake-shoe pressure of 6000 lb. on each of the 8 wheels. If the coefficient of brake friction is .15, how far will the car move before coming to rest?

Problem 264. In the above problem the kinetic energy of rotation of the wheels, the axle friction, and the rolling friction have been neglected. The coefficient of friction for the journals is .002, that for rolling friction is .02. Each pair of wheels and axle has a mass of 45 and a moment of inertia with respect to the axis of rotation of 37. The diameter of the wheels is 32 in. and the radius of the axles is 2½ in. Compute the distance the car in the preceding problem will go before coming to rest. Compare the results.

Problem 265. A 30-ton car is running at the rate of 70 mi. per hour on a level track when the power is turned off and brakes applied so that the wheels are just about to slip on the rails. If the coefficient of friction of sliding between wheels and rails is .20, how far will the car go before coming to rest?

Problem 266. A 75-ton locomotive going at the rate of 50 mi. per hour is to be stopped by brake friction within 2000 ft. If the coefficient of friction is .25, what must be the normal brake-shoe pressure?

Problem 267. A 75-ton locomotive has its entire weight carried by five pairs of drivers (radius 3 ft.). The mass of one pair of drivers is 271 and the moment of inertia is 1830. If, when moving with a velocity of 50 mi. per hour, brakes are applied so that slipping on the rails is impending, how far will it go before being stopped? The coefficient of friction between the wheels and rails is .20.

168. Train Resistance. — The resistance offered by a train depends upon a number of conditions, such as velocity, acceleration, the condition of track, number of cars, curves, resistance of the air, and grades. No law of resistance can be worked out from a theoretical consideration, because of the uncertainty of the influence of the various factors involved. Formulæ have been developed from the results of tests; the most important of these are given below.

Let R represent the resistance in pounds and v the velocity in miles per hour. W. F. M. Goss has found that the resistance may be expressed as

$$R = .0003\,(L + 347)\,v^2,$$

where L is the length of the train in feet (see *Engineering Record*, May 25, 1907).

The Baldwin Locomotive Works have derived the formula

$$R = 3 + \frac{v}{6}$$

as the relation between the resistance and velocity. When all factors are considered, this becomes

$$R = 3 + \frac{v}{6} + .3788\,(t) \pm .5682\,(c) + .01265\,(a)^2,$$

where t = grade in feet per mile, c the degree of curvature of the track, and a the rate of increase of speed in miles per hour in a run of one mile.

To get the total resistance it is necessary to include, in addition to the above factors, the friction of the locomotive and tender. This is given by Holmes (see Kent's "Hand Book") as

$$R_1 = [12 + .3\,(v - 10)\,\text{W}],$$

where W is the weight of the engine and tender in pounds and R_1 the resistance in pounds due to friction.

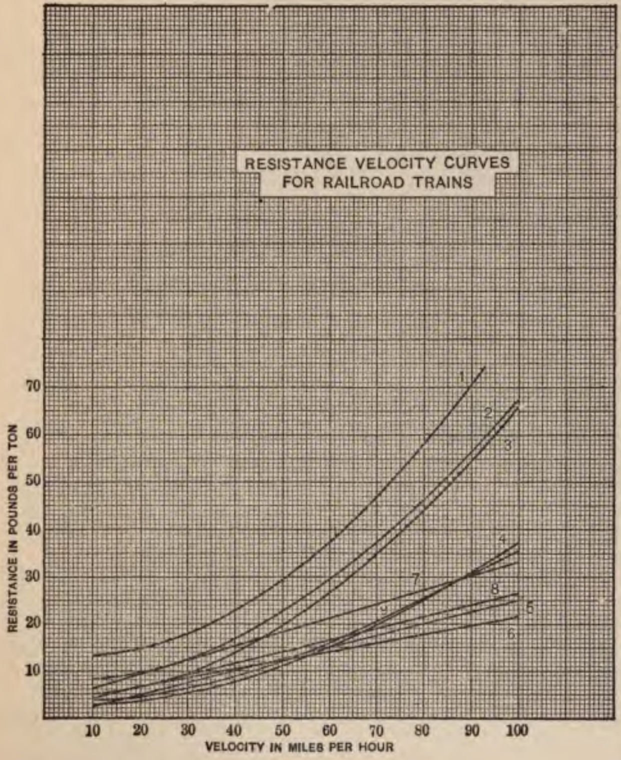


FIG. 197

Other formulæ derived as the result of experiments are shown graphically in Fig. 197.

The formulæ themselves are as follows (see *Engineering*, July 26, 1907):

CURVE NUMBER	FORMULA	AUTHORITY
1	$R = 12 + \frac{v^2}{144}$	Clark
2	$R = 8 + \frac{v^2}{171}$	Clark
3	$R = 4.48 + \frac{v^2}{162}$	Wellington
4	$R = 3 + \frac{v^2}{290}$	Deeley
5	$R = .2497\, v$	Laboriette
6	$R = 3.36 + .1867\, v$	Baldwin Company
7	$R = 4.48 + 284\, v$	Lundie
8	$R = 2 + .24\, v$	Sinclair
9	$R = 2.5 + \frac{v^{\frac{5}{3}}}{65.82}$	Aspinal

It is evident that these formulæ do not agree as closely as one would wish. The difference must be due chiefly to the different conditions under which the tests were made. These conditions should be taken into account in any application of the formulæ to special cases.

CHAPTER XV

IMPACT

169. Definitions.—When two bodies that are approaching each other collide, they are said to be subjected to *impact*. If their motion is along the line joining their centers of gravity, the collision is designated as *direct central impact*. If they are moving along parallel lines, not the common gravity line, the impact is known as *direct eccentric impact*. When the collision differs from either of the above forms, the impact is known as *oblique impact*.

The phenomena of impact may be best studied by considering the two bodies somewhat elastic. Suppose for simplicity that they are two spheres, M_1 and M_2, Fig. 198, and that they are moving in opposite directions with velocities v_1 and v_2 and that the impact is central. In Fig. 198 (*a*) they are shown at the instant when contact first takes place, and in Fig. 198 (*b*) they are shown some time after first contact when each has been deformed somewhat by the pressure of the other. The dotted lines indicate the original spherical form and the full lines the actual form of the deformed spheres. When the spheres first touch, the pressure P between them is zero, but as each one compresses the other, the pressure P increases until it becomes a maximum. The compression of the spheres is indicated in the figure by d_1 and d_2. We shall designate the time during which the bodies are being compressed as the *period of deformation.*

After the compression has reached its maximum value the bodies, if they be partially elastic, begin to separate and to regain their original form. The common pressure P decreases and becomes zero, if the bodies are sufficiently elastic so that they finally separate. We shall designate this period of separation as the *period of restitution*, and the velocities of separation as v_1' and v_2'.

If the bodies are entirely *inelastic*, there will be no restitution. They will, in that case, remain in contact just as they are when the pressure between them is a maximum and will move on with a common velocity V.

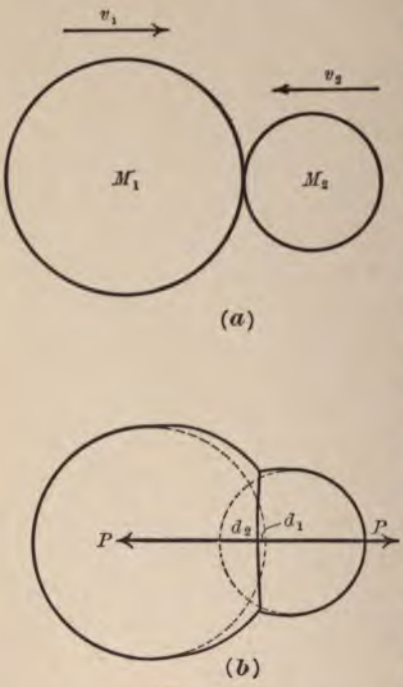


Fig. 198.

170. Direct Central Impact, Inelastic.—When the bodies meet in direct central impact, separation will take place along the line joining the centers of gravity. Let T be the time from the first contact up to the time of maximum pressure, that is, the *time of deformation*, and T_1 the time from first contact up to the time of separation. Then $T_1 - T$ represents *the time of restitution.* We have, from Art. 72, $dv = a \cdot dt$. Considering the motion of M_1 during the period of deformation, we have $dv_1 = a_1 dt$ and $a_1 = -\frac{P}{M_1}$,

so that
$$\int_{v_1}^{V} dv_1 = -\frac{1}{M_1}\int_0^T P dt,$$

or
$$M_1(V - v_1) = -\int_0^T P dt.$$

In a similar way, remembering that if v_1 is positive v_2 is negative,
$$M_2(V + v_2) = \int_0^T P dt.$$

The two integrals $\int_0 P dt$ on the right-hand side of the preceding equations cannot be determined since we do not know in general how the pressure P varies with the time; we do know, however, that they are equal term for term, so that we may eliminate them. We have, then,
$$-M_1(V - v_1) = M_2(V + v_2),$$

or
$$V = \frac{M_1 v_1 - M_2 v_2}{M_1 + M_2}$$

If the bodies were moving in the direction of M_1 and $v_1 > v_2$, we should have both v_1 and v_2 positive, a_1 negative, and a_2 positive. Then
$$V = \frac{M_1 v_1 + M_2 v_2}{M_1 + M_2}.$$

This is also the value for V if both bodies are moving in the direction v_2 and $v_2 > v_1$.

If the bodies are inelastic, they will both move with the velocity V, and there will be no separation. Suppose the bodies to be two lead balls, and let $G_1 = 10$ lb., $G_2 = 25$ lb., $v_1 = 40$ ft. per second, and $v_2 = 60$ ft. per second. Then $V = 54.28$ ft. per second if the bodies are moving in the same direction, and $V = 33.3$ ft. per second if they move in opposite directions.

The *energy lost* in direct central impact of *inelastic* bodies may be found by subtracting the kinetic energy of M_1 and M_2, when the common velocity is V, from the kinetic energy of the two bodies at the time of first contact. The kinetic energy of M_1 before impact is $E_1 = \frac{1}{2} M_1 v_1^2$, and that of M_2 is $E_2 = \frac{1}{2} M_2 v_2^2$, so that the total energy before impact is $\frac{1}{2} M_1 v_1^2 + \frac{1}{2} M_2 v_2^2$. The kinetic energy after impact is $\frac{1}{2}(M_1 + M_2) V^2$, so that the loss of kinetic energy is

$$\tfrac{1}{2} M_1 v_1^2 + \tfrac{1}{2} M_2 v_2^2 - \tfrac{1}{2} (M_1 + M_2)\ V^2,$$

and this equals
$$\frac{M_1 M_2 (v_1 - v_2)^2}{2(M_1 + M_2)}$$

if the bodies move in the same direction, or

$$\frac{M_1 M_2 (v_1 + v_2)^2}{2(M_1 + M_2)}$$

if they move in opposite directions.

This energy is used up in deforming the bodies and in raising their temperatures. The kinetic energy remaining may be written

$$\tfrac{1}{2}(M_1 + M_2) V^2 = \tfrac{1}{2} \frac{(M_1 v_1 + M_2 v_2)^2}{M_1 + M_2}$$

when the bodies move in the same direction.

In the above example of the lead balls, we find that the kinetic energy lost due to impact is 40.3 ft.-lb. when the bodies move in the same direction, and 1109 ft.-lb. when they move in opposite directions.

When the bodies move toward each other and $M_1v_1 = M_2v_2$, the final velocity V is zero and the kinetic energy lost is $\frac{1}{2}M_1v_1^2 + \frac{1}{2}M_2v_2^2$. If the masses are equal, $V = \frac{v_1 - v_2}{2}$ and the kinetic energy lost is

$$\frac{M}{2}\frac{(v_1 + v_2)^2}{2}.$$

If M_2 is infinite as compared with M_1, and v_2 is zero, the final velocity V is zero, and the kinetic energy lost is $\frac{M_1v_1^2}{2}$.

Problem 268. A lead sphere whose radius is 2 in. strikes a large mass of cast iron after falling freely from rest through a distance of 100 ft. What is its final velocity? What is the loss of kinetic energy?

Problem 269. A 10-lb. lead sphere is at rest when it is acted upon by another lead sphere, whose radius is 3 in., in direct central impact. The velocity of the latter sphere is 20 ft. per second. What is the common velocity of the two spheres and what is the loss of kinetic energy due to impact?

171. Direct Central Impact, Elastic. — If the impact is not too severe, elastic or partially elastic bodies tend to regain their original shape after the deformation has reached a maximum and finally separate if they possess sufficient elasticity. Using the notation of Art. 169, we have, for the period of restitution, if R is the force of restitution,

$$M_1\int_V^{v'_1} dv_1 = -\int_T^{T_1} R\,dt, \quad \text{and for } M_2,$$

$$M_2\int_V^{v'_2} dv_2 = \int_T^{T_1} R\,dt,$$

so that $$M_1(v'_1 - V) = -\int_T^{T_1} R\,dt,$$

and $$M_2(v'_2 - V) = \int_T^{T_1} R\,dt$$

The value of the integral $\int P\,dt$ during deformation will not in general be the same as its value during restitution. Call the ratio of $\int_T^{T_1} R\,dt$ to $\int_0^T P\,dt$, e. This value, which, is called the *coefficient of restitution*, is constant for a given material. It is unity for perfectly elastic substances, zero for non-elastic substances, and some intermediate value for the imperfectly elastic materials with which the engineer is usually concerned. The following values of e have been determined: for steel, $e = .55$, for cast iron, $e = 1$, nearly; for wood, $e = 0$, nearly.

From the above definition of e, it is seen at once that we may write

$$e_1 = \frac{v'_1 - V}{V - v_1}, \text{ and } e_2 = \frac{v'_2 - V}{V - v_1},$$

and these equations enable one to determine e experimentally.

Since $$\int_T^{T_1} R\,dt = e\int_0^T P\,dt,$$

we may write

$$M_1(v'_1 - V) = e_1 M_1(V - v_1),$$

$$M_2(v_2' - V) = e_2M_2(V - v_2),$$

so that $$v_1' = V(1 + e_1) - e_1v_1,$$

and $$v_2' = V(1 + e_2) - e_2v_2,$$

where $$V = \frac{M_1v_1 + M_2v_2}{M_1 + M_2},$$

if the bodies are moving in the same direction, and

$$\frac{M_1v_1 - M_2v_2}{M_1 + M_2}$$

if they are moving in opposite directions. If the bodies are of the same material, $e_1 = e_2 = e$. Then from the above equation it is seen that

$$v_1' - v_2' = - e(v_1 - v_2),$$

and the energy lost in impact is

$$(1 - e^2)\frac{M_1M_2}{2(M_1 + M_2)}(v_1 - v_2)^2.$$

If the bodies are perfectly elastic, so that $e = 1$, the loss of energy is zero.

Problem 270. The student should show that for any impact

$$M_1v_1 + M_2v_2 = M_1v_1' + M_2v_2';$$

that is, the sum of the momenta before impact equals the sum of the momenta after impact.

Problem 271. Two perfectly elastic bodies, having equal velocities in opposite directions, meet in direct central impact. What must be the relation of their masses so that they will be reduced to rest?

Problem 272. If the bodies are perfectly elastic and $M_1 = M_2$, show that $v_1' = v_2$ and $v_2' = v_1$.

Problem 273. If a ball M_1 of a certain material falls upon a large mass M_2 of the same material from a height h and rebounds to a height h_1, show that $$e=\sqrt{\frac{h_1}{h}}.$$

In this case $M_2=\infty$, $v_2=0$, and $V=0$.

Problem 274. A 20-ton car having a velocity of 40 mi. per hour collides with a 30-ton car having a velocity of 60 mi. per hour in the opposite direction. Both are destroyed. What is the loss of kinetic energy?

172. Elasticity of Material. — All materials of engineering are imperfectly elastic. Some, however, show almost perfect elasticity for stresses that are rather low. This has been expressed by saying that all materials have a limit (elastic limit) beyond which if the stress be increased the material will be imperfectly elastic. Within the *limit of elasticity*, *stress* is proportional to the *deformation* produced. Let the total stress in tension or compression be P, and the stress, in pounds per square inch of cross section, be f, also let d be the deformation caused by P and λ the deformation per inch of length. Within the limit of elasticity of the material the ratio $\frac{f}{\lambda}$ is a constant, and since $f=\frac{P}{F}$, and $\lambda=\frac{d}{l}$, when F is the area of cross section and l is the length of the material, it may be written $\frac{Pl}{Fd}$. This constant is called the *modulus of elasticity* of the material; it is usually represented by E, so that

$$E=\frac{f}{\lambda}=\frac{Pl}{Fd}$$

for tension or compression. For steel E has been found to be 30,000,000 lb. per square inch.

173. Impact of Imperfectly Elastic Bodies. — Let d_1 be the compression of M_1 due to the impact, and d_2 the compression of M_2. If P_m is the pressure between the two bodies when the compression is greatest, the average force acting may be represented by $\frac{P_m}{2}$, if the limit of elasticity of the material is not passed. The work done on the two bodies is $\frac{P_m}{2}(d_1+d_2)$, and this should equal the energy lost during compression; then

$$\frac{P_m}{2}(d_1+d_2)=\frac{M_1M_2(v_1-v_2)^2}{2(M_1+M_2)},$$

so that
$$P_m=\frac{M_1M_2(v_1-v_2)^2}{(M_1+M_2)(d_1+d_2)}.$$

From the preceding article we know that $d_1=\frac{P_ml_1}{F_1E_1}$ and $d_2=\frac{P_ml_2}{F_2E_2}$, where l_1 and l_2 are the lengths of the masses M_1 and M_2 (considered prismatic), F_1 and F_2 the areas of cross section, and E_1 and E_2 the moduli of elasticity. The sum d_1+d_2 may then be represented by

$$P_m\left(\frac{l_1}{F_1E_1}+\frac{l_2}{F_2E_2}\right).$$

For convenience let $\frac{F_1E_1}{l_1}=H_1$ and $\frac{F_2E_2}{l_2}=H_2$ (H_1 and H_2 may be considered as representing the hardness).

Then
$$d_1+d_2=P_m\left(\frac{H_2+H_1}{H_2H_1}\right)$$

and
$$P_m=(v_1-v_2)\sqrt{\frac{M_1M_2}{M_1+M_2}\left(\frac{H_2H_1}{H_2+H_1}\right)}.$$

Problem 275. Suppose a 100-lb. steel hammer strikes an immovable cast-iron plate with a velocity of 25 ft. per second. The hammer has a face of 3 sq. in. area and a length of 6 in.; the plate has the same area and a thickness of 2 in. If the modulus of elasticity of steel is 30,000,000 and of cast iron is 15,000,000, find the greatest pressure between the two bodies due to the impact.

Under the assumptions $v_2 = 0$ and $M_2 = \infty$,

$$M_1 = \frac{100}{32.2},\ H_2 = 22{,}500{,}000,\ H_1 = 15{,}000{,}000.$$

Then $P_m = 132{,}750$ lb.

Problem 276. Let the mass of the ram of a pile driver be M_1 and let h be the height of fall. Let M_2 be the mass of the pile and s its penetration under a blow. If d_1 is the compression of the hammer and d_2 the compression of the pile, the work equation becomes

$$P_m s + \frac{P_m}{2}(d_1 + d_2) = G_1 h,$$

and

$$P_m = \frac{G_1 h}{s + \frac{1}{2}(d_1 + d_2)}.$$

It is required to find the load that the pile will carry. Since $v_1 = \sqrt{2\,gh}$ and $v_2 = 0$, we may write

$$d_1 + d_2 = \sqrt{\frac{2\,M_1 M_2 gh}{M_1 + M_2}\left(\frac{H_2 + H_1}{H_2 H_1}\right)},$$

and then

$$P_m = \frac{G_1 h}{s + \sqrt{\dfrac{M_1 M_2 gh}{2(M_1 + M_2)}\left(\dfrac{H_2 + H_1}{H_2 H_1}\right)}}.$$

The load P_m that the pile will carry is found by measuring the penetration s for the last blow. It is customary to use a factor of safety, as was explained in Art. 137.

Problem 277. A wooden pile whose cross section is 1.5 sq. ft., and whose length is 30 ft., is driven by a steel hammer of 2000 lb. weight falling a distance of 20 ft. The penetration at the last blow is observed to be $\frac{1}{2}$ in. What load will the pile carry, using a factor of safety of 6?

Assume the weight of the pile as 1800 lb., the modulus of elasticity of timber as 1,500,000, and of steel as 30,000,000, the face of the hammer 2 sq. ft., and its length 2 ft.

An approximate formula may be obtained by noting that l_2 is large as compared with l_1 and E_2 is small as compared with E_1, so that H_1 is large compared with H_2, thus making

$$\frac{H_2 + H_1}{H_2H_1} = \frac{\frac{H_2}{H_1} + 1}{H_2} = \frac{1}{H_2} \text{ (approximately).}$$

Substituting in the expression for P_m, we have

$$P_m = \frac{G_1h}{s + \sqrt{\frac{M_1M_2gh}{(M_1 + M_2)H_2}}}.$$

Problem 278. The student should solve Problem 277, using this approximate formula and compare the result with that already obtained.

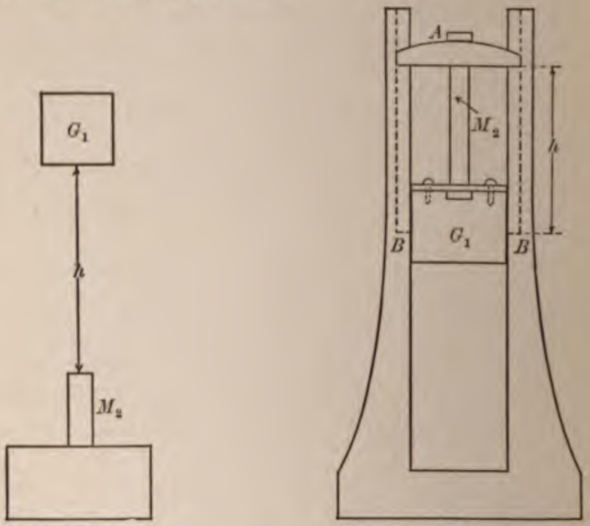
Problem 279. Compare the results obtained in problems 277 and 278 with those obtained by using formulas of Art. 137.

174. Impact Tension and Impact Compression. — Figure 199 (*a*) represents a mass M_2 subjected to impact from the mass M_1 falling from rest through a height h. The mass M_2 is compressed by the impact. Figure 199 (*b*) represents the body M_2 as subjected to impact in tension, the mass in this case being a rod having G_1 attached to one end and the other end attached to a crosshead A. The rod, crosshead, and weight fall freely together through the distance h until A strikes the stops at B, when one end of the rod suddenly comes to rest and the weight G_1 causes tension in the rod due to impact.

Suppose v_1 represents the velocity of M_1 when impact occurs and l_1 its length, whether it be a tension or compression piece. Let V be the common velocity of the bodies at the time of greatest pressure. Then $V=\frac{M_1 v_1}{M_1+M_2}$ (see Art. 170), and the kinetic energy necessary to bring the two bodies to rest is given by the expression,

$$\tfrac{1}{2}(M_1+M_2)V^2=\frac{M_1^2 v_1^2}{2(M_1+M_2)}=\frac{M_1^2 gh}{M_1+M_2},$$

where h is the height of fall.



(*a*) (*b*)

Fig. 199

This energy is used in stretching or compressing M_2, if we neglect the work done on G_1, which is supposed small in comparison. Let d_2 be the deformation of M_2 when the pressure between the two bodies is greatest, and let the average pressure between them be $\frac{P_m}{2}$, as before. Then the work done on the bar may also be expressed as $\frac{P_m}{2} \times d_2$ and

$$\frac{P_m}{2} d_2 = \frac{E_2F_2d_2^2}{2l_2} = \frac{M_1^2gh}{M_1 + M_2},$$

where F_2 is the cross section, l_2 the length, and E_2 the modulus of elasticity of M_2. We may, therefore, write

$$d_2 = \sqrt{\frac{M_1^2}{M_1 + M_2}\,\frac{2gl_2h}{E_2F_2}}$$

as the elongation or compression of M_2 due to the impact.

Problem 280. A weight of 500 lb. falls through a distance of 2 ft. in such a way as to put a 1-in. round steel rod in tension. If the rod is 18 in. long, what will be the elongation due to the impact?

In this case $M_1 = \frac{500}{32.2}$, $M_2 = \frac{3.8}{32.2}$, $l_2 = 18$, $h = 24$, $E_2 = 30{,}000{,}000$, and $F_2 = \frac{11}{14}$ sq. in.

Problem 281. A cylindrical piece of steel 1 in. high and 1 in. in diameter is subjected to compression by a weight of 20 lb. falling through a distance of 1 in. How much will it be compressed?

If we substitute for P_m its equivalent f_2F_2 (Art. 172), where f_2 represents the stress in M_2 in pounds per square inch of cross section, we may write

$$d_2 = \frac{f_2l_2}{E_2},$$

so that

$$\frac{f_2^2l_2^2}{E_2^2} = \frac{M_1^2}{M_1 + M_2}\,\frac{2gl_2h}{E_2F_2},$$

which may be written

$$\frac{f_2^2}{2E_2}(F_2l_2) = \frac{M_1^2gh}{M_1 + M_2}.$$

Since F_2l represents the volume of M_2, we have

$$h = \frac{f_2^2}{2E_2}(\text{vol.})^2\,\frac{M_1 + M_2}{gM_1^2}.$$

This equation represents the relation between the height of fall of M_1 and the stress f_2 produced by such fall.

Problem 282. In the preceding problem what stress (pounds per square inch) was caused in the cylindrical block by the fall of the 20-lb. weight?

Problem 283. The safe stress in structural steel for moving loads, impact loads, is usually taken as 12,500 lb. per square inch (value of f_2). Through what height might a 300-lb. weight fall so as to produce tension in a 1-in. steel round bar, 10 ft. long, without exceeding the safe stress?

Problem 284. Two steel tension rods in a bridge, each of 2 sq. in. in cross section and 20 ft. long, carry the effect of the impact of a loaded wagon as one wheel rolls over a stone 1 in. high. The weight on the wheel is 2000 lb. What stress is introduced in the tension rods?

NOTE. For the strength of metals under impact the student is referred to the work of W. K. Hatt, Am. Soc. "Testing Materials," Vol. IV, p. 282.

175. Direct Eccentric Impact. — The impact is said to be *direct eccentric* when the line of motion does not coincide with the line joining the centers of gravity of the two bodies (see Fig. 200). Suppose M_1 at rest and that it is acted upon by M_2 moving with a velocity v_2 in the direction shown, and that P_m is the force exerted by M_2 on M_1. We shall first show that the motion imparted to M_1 may

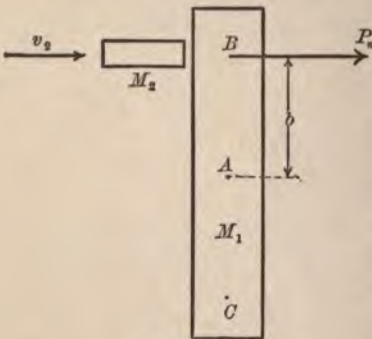


FIG. 200

be considered a rotation about its center of gravity A combined with a translation of that center of gravity.

Introduce at a point (c), distant b below A, two equal and opposite forces equal to $\frac{P_m}{2}$ and parallel to P_m. The introduction of these forces does not change the state of motion of the body. Consider one half of P_m acting at B with $\frac{1}{2}P_m$ acting at C in the same direction. These two forces are equivalent to a single force P_m acting at A in the direction P_m. The remaining $\frac{1}{2}$ of P_m at B with the $\frac{1}{2}P_m$ at C form a couple, of moment $P_m b$, which tends to produce rotation about the center of gravity A. The motion, then, imparted to M_1 may be considered as consisting of a *translation* and a *rotation.*

Considering the motion of M_1 and calling P the variable pressure between M_1 and M_2, we may write

$$M_1\int_0^V dv = \int_0^T Pdt \qquad \text{(see Art. 170)},$$

and since rotation also occurs, and $d\omega = \theta dt = \frac{Pb}{M_1 k^2_A}dt$ (see Art. 103), we may also write

$$k^2_A M_1\int_0^{\omega_1} d\omega = b\int_0^T Pdt.$$

These equations become upon integration

$$M_1 V = \int_0^T Pdt,$$

$$\frac{k^2_A}{b}M_1\omega_1 = \int_0^T Pdt.$$

Considering now the motion of M_2, we may write

$$M_2\int_{v_2}^V dv = -\int_0^T Pdt,$$

which gives

$$M_2(V - v_2) = -\int_0^T P dt.$$

Eliminating $\int_0^T P dt$ from these equations, we have

$$bV = k'^2_A \omega_1,$$

and

$$M_1 V = M_2(v_2 - V),$$

and since

$$V = \omega_1 b,$$

these equations are sufficient to determine V and ω_1.

Problem 285. If the bodies are both inelastic, find the kinetic energy lost in direct eccentric impact.

Problem 286. Suppose M_1 to be a bar of steel ½ in. in diameter and 2 ft. long, and suppose M_2 to be a hammer weighing 2 lb. and that its velocity at the time of impact is 20 ft. per second, find V and ω_1 if the hammer strikes 10 in. from the center of the rod.

Problem 287. Let M_1 be a square stick of timber 4″ × 4″ × 10′ and let M_2 be a 10-lb. hammer having a velocity at the time of impact of 10 ft. per second. If the impact takes place 4 ft. from the center, find V and ω_1.

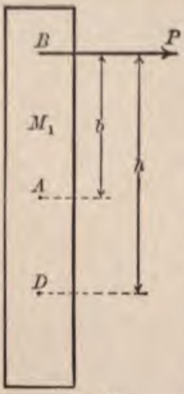


FIG. 201

176. Center of Percussion. — We have seen that the motion of M_1 in direct eccentric impact may be considered as being made up of a rotation combined with a translation. As a matter of fact, however, the motion that actually occurs is a rotation about an instantaneous center. We shall now find such center of rotation, called *center of percussion* (see Art. 106).

Let M_1, Fig. 201, be the body under consideration and let P be the force caused by the impact and h the distance of P from the center of rotation D. We have shown in the preceding article that

$$V = \frac{k_A^2 \omega_1}{b},$$

when V and ω_1 are the velocities of the body at time of greatest pressure. Now $V = \omega_1(h - b)$, since D is momentarily at rest. Therefore

$$\omega_1(h - b) = \frac{k_A^2 \omega_1}{b},$$

so that
$$(h - b)b = k_A^2.$$

The quantities b and k_A^2 are usually known, so that h may readily be computed.

Problem 288. A right circular cone of steel, radius of whose base is 6 in. and altitude 6 in., is supported as a pendulum by an axis through its vertex parallel to the base. It is struck with a 3-lb. hammer with a velocity of 10 ft. per second, at the center of percussion. Find V and ω at time of greatest pressure.

Problem 289. A man strikes a blow with a steel rod $1\frac{1}{2}$ in. in diameter and 4 ft. long, by holding the rod in the hand and striking the farther end against a stone in such a way as to cause the rod to be under flexure. Where should he grasp the rod in order that he may receive no shock?

177. Oblique Impact of Body against Smooth Plane.

— Let M (Fig. 202) be a sphere moving toward the plane indicated with a velocity at impact of v, the direction of motion making an angle α with the vertical to the plane. After im-

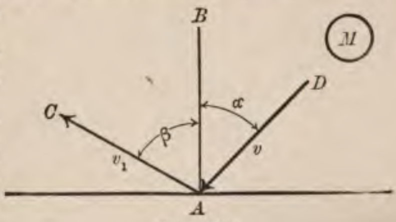


FIG. 202

pact the body M rebounds with a velocity v_1 in the direction, making an angle β with the vertical AB. Since the plane is considered smooth, the effect of the impact will be all in the direction of AB and the impulsive force after impact will be e times what it was before impact.

Summing horizontal and vertical components of the velocities, we have

$$v_1 \sin\beta = v \sin\alpha;$$

$$v_1 \cos\beta = ev \cos\alpha.$$

Dividing, we have

$$\tan\beta = \frac{1}{e}\tan\alpha,$$

and squaring and adding,

$$v_1^2 = v^2(\sin^2\alpha + e^2\cos^2\alpha).$$

So that if α and e are known, v_1 and β may be determined.

If the body is perfectly elastic, $e = 1$, $\beta = \alpha$, and $v_1 = v$. If the body is inelastic, so that $e = 0$, $\beta = \frac{\pi}{2}$, $v_1 = v \sin\alpha$, it then moves along the plane with a velocity $v \sin\alpha$.

178. Impact of Rotating Bodies. — Suppose two bodies M_2 and M_1 revolve about two parallel axes O and O_1 (Fig. 203) in such a way that impact occurs at a point along the line DE. Let the point at which impact occurs be distant r_2 from O and r_1 from O_1. It is evident that the kinetic energy of M_2 is $\frac{1}{2} I_0 \omega_2^2$ and that this is equal to the kinetic

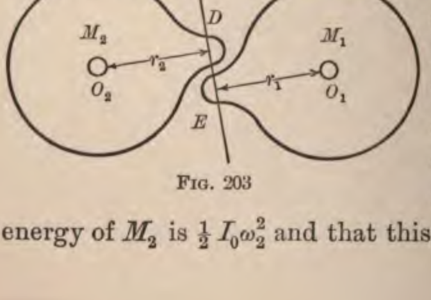


FIG. 203

energy of an equivalent mass M' situated at a distance r_2 from O, since $\frac{1}{2}M'v^2 = \frac{1}{2}M'r_2^2\omega_2^2$. Equating these values for kinetic energy, we have for the equivalent mass M' the value $\frac{M_2k_0^2}{r_2^2}$. Likewise the equivalent mass of M_1 at the point of impact is $\frac{M_1k_{0_1}^2}{r_1^2}$. The impact of the two rotating bodies M_2 and M_1, then, may be considered by considering the impact of their equivalent masses along the line DE. From Art. 171, we have

$$v_1' = V(1 + e_1) - e_1v_1$$

and

$$v_2' = V(1 + e_2) - e_2v_2,$$

where $V = \frac{M_1v_1 + M_2v_2}{M_1 + M_2}$, if the bodies are moving in the same direction, and a similar expression with v_2, say, negative, if they are moving in opposite directions.

Let ω_2 be the angular velocity of M_2 before impact.

Let ω_1 be the angular velocity of M_1 before impact.

Let ω_2' be the angular velocity of M_2 after impact.

Let ω_1' be the angular velocity of M_1 after impact.

Then we may write

$$\omega_2r_2 = v_2,\ \omega_1r_1 = v_1,\ \omega_2'r_2 = v_2',\ \omega_1'r_1 = v_1',$$

so that

$$\omega_1' = r_2\left[\frac{M_1\omega_1r_2k_{0_1}^2 + M_2\omega_2r_1k_0^2}{M_1k_{0_1}^2r_2^2 + M_2k_0^2r_1^2}\right](1 + e_1) - e_1\omega_1,$$

$$\omega_2' = r_1\left[\frac{M_1\omega_1r_2k_{0_1}^2 + M_2\omega_2r_1k_0^2}{M_1k_{0_1}^2r_2^2 + M_2k_0^2r_1^2}\right](1 + e_2) - e_2\omega_2.$$

These equations may be put in the form

$$\omega_1' = \omega_1 - r_1(r_1\omega_1 - r_2\omega_2)\,(1 + e_1)\,\frac{I_2}{I_1 r_2^2 + I_2 r_1^2},$$

$$\omega_2' = \omega_2 - r_2\,(r_1\omega_1 - r_2\omega_2)(1 + e_2)\,\frac{I_1}{I_1 r_2^2 + I_2 r_1^2}.$$

Problem 290. Suppose the moment of inertia of M_1 is 3000 and its angular velocity before impact one radian per second; that of M_2

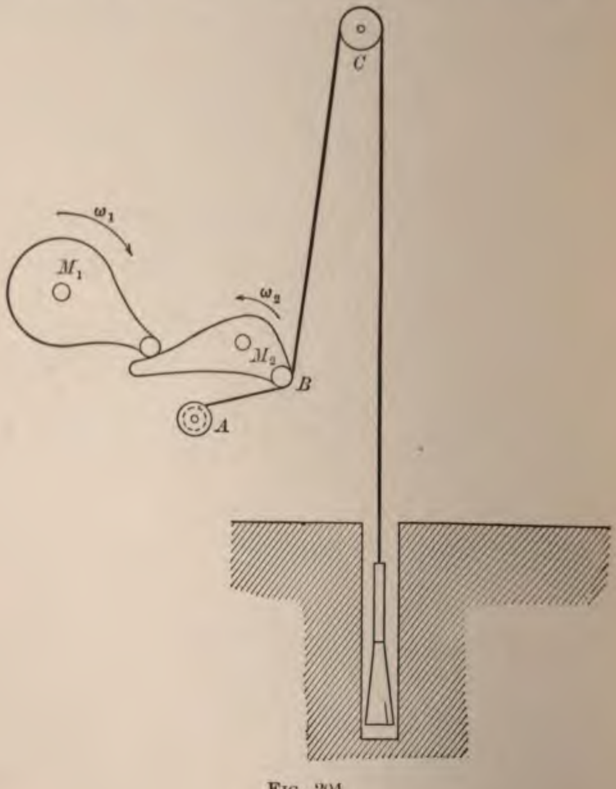


FIG. 204

15,000 and its angular velocity zero. Let $r_1 = 2$ ft. and $r_2 = 3$ ft. and $e_1 = e_2 = 0$.

Then $\omega_1' = .311$ radian per second.

$\omega_2' = -.207$ radian per second.

The kinetic energy lost due to the impact is

$$\frac{I_1\omega_1^2}{2} - \frac{I_1\omega_1'^2}{2} - \frac{I_2\omega_2'^2}{2} = 1034 \text{ ft.-lb.}$$

Problem 291. A well drill is shown in principle in Fig. 204. The drill is supported by a cable that passes over a pulley C and is attached to a friction drum A. When A is held, the drill is raised by the operation of M_1 and M_2. Suppose that I is 300 and $\omega_1 = 3$ radians per second; $I_2 = 200$ and $\omega_2 = 0$; $r_1 = 2$ ft. and $r_2 = 6$ ft. Assume $e_1 = e_2 = \frac{1}{2}$. Find ω_1' and ω_2'. What kinetic energy is lost due to each impact?

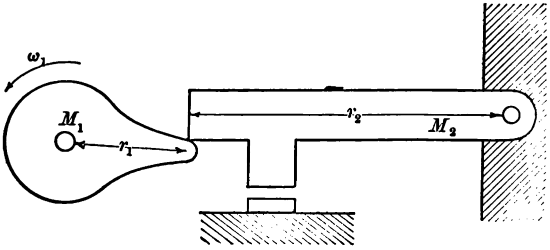


FIG. 205

Problem 292. The moment of inertia of the trip hammer M_2, illustrated in principle in Fig. 205, is 100,000; that of M_1 is 60,000. If $r_1 = 3$ ft., $r_2 = 10$ ft., $\omega_1 = 2$ radians per second, $\omega_2 = 0$, and $e_1 = e_2 = \frac{1}{2}$, find ω_1' and ω_2'. What is the kinetic energy lost due to each impact? What is the kinetic energy of the hammer?

APPENDIX I

HYPERBOLIC FUNCTIONS

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

1

x	Cosh x	Sinh x	x	Cosh x	Sinh x
0.01	1.0000500	0.0100002	0.51	1.1328934	0.5323978
.02	.0002000	.0200013	.52	.1382741	.5437536
.03	.0004500	.0300045	.53	.1437686	.5551637
.04	.0008000	.0400107	.54	.1493776	.5666292
.05	.0012503	.0500208	.55	.1551014	.5781516
.06	.0018006	.0600360	.56	.1609408	.5897317
.07	.0024510	.0700572	.57	.1668962	.6013708
.08	.0032017	.0800854	.58	.1729685	.6130701
.09	.0040527	.0901215	.59	.1791579	.6248305
.10	.0050042	.1001668	.60	.1854652	.6366536
.11	.0060561	.1102220	.61	.1918912	.6485402
.12	.0072086	.1202882	.62	.1984363	.6604917
.13	.0084618	.1303664	.63	.2051013	.6725093
.14	.0098161	.1404578	.64	.2118867	.6845942
.15	.0112711	.1505631	.65	.2187933	.6967475
.16	.0128274	.1606835	.66	.2258219	.7089704
.17	.0144849	.1708200	.67	.2329730	.7212643
.18	.0162438	.1809735	.68	.2402474	.7336303
.19	.0181044	.1911452	.69	.2476458	.7460697
.20	.0200668	.2013360	.70	.2551690	.7585837
.21	.0221311	.2115469	.71	.2628178	.7711735
.22	.0242977	.2217790	.72	.2705927	.7838405
.23	.0265668	.2320333	.73	.2784948	.7965858
.24	.0289384	.2423107	.74	.2865248	.8094107
.25	.0314132	.2526122	.75	.2946833	.8223167
.26	.0339908	.2629393	.76	.3029713	.8353049
.27	.0366720	.2732925	.77	.3113896	.8483766
.28	.0394568	.2836731	.78	.3199392	.8615330
.29	.0423456	.2940819	.79	.3286206	.8747758
.30	.0453385	.3045203	.80	.3374349	.8881060
.31	.0484361	.3149891	.81	.3463831	.9015249
.32	.0516384	.3254894	.82	.3554658	.9150342
.33	.0549460	.3360222	.83	.3646840	.9286347
.34	.0583590	.3465886	.84	.3740388	.9423282
.35	.0618778	.3571898	.85	.3835309	.9561160
.36	.0655029	.3678265	.86	.3931614	.9699993
.37	.0692345	.3785001	.87	.4029312	.9839796
.38	.0730730	.3892116	.88	.4128413	0.9980584
.39	.0770189	.3999619	.89	.4228927	1.0122369
.40	.0810724	.4107523	.90	.4330864	.0265167
.41	.0852341	.4215838	.91	.4434234	.0408991
.42	.0895042	.4324574	.92	.4539048	.0553856
.43	.0938888	.4433742	.93	.4645315	.0699777
.44	.0983718	.4543354	.94	.4753046	.0846768
.45	.1029702	.4653420	.95	.4862254	.0994843
.46	.1076788	.4763952	.96	.4972947	.1144018
.47	.1124983	.4874959	.97	.5085137	.1294307
.48	.1174289	.4986455	.98	.5198837	.1445726
.49	.1224712	.5098450	.99	.5314057	.1598288
0.50	1.1276260	0.5210953	1.00	1.5430806	1.1752012

2

x	Cosh x	Sinh x	x	Cosh x	Sinh x
1.01	1.5549100	1.1906910	1.51	.2338704	2.1529104
1.02	.5668948	.2062999	1.52	.3954686	.1767566
1.03	.5790365	.2220294	1.53	.4173563	.2008206
1.04	.5913358	.2378812	1.54	.4394857	.2251046
1.05	.6037945	.2538567	1.55	.4618591	.2496111
1.06	.6164134	.2699576	1.56	.4844787	.2743426
1.07	.6291940	.2861855	1.57	.5073467	.2993014
1.08	.6421375	.3025420	1.58	.5304654	.3244903
1.09	.6552453	.3190288	1.59	.5538373	.3499117
1.10	.6685186	.3356474	1.60	.5774645	.3755679
1.11	.6819587	.3523997	1.61	.6013494	.4014618
1.12	.7005670	.3642872	1.62	.6254945	.4275958
1.13	.7093449	.3863116	1.63	.6499021	.4539726
1.14	.7232938	.4034746	1.64	.6745748	.4805947
1.15	.7374148	.4207781	1.65	.6995149	.5074650
1.16	.7517098	.4382235	1.66	.7247249	.5345859
1.17	.7661798	.4558128	1.67	.7502074	.5619603
1.18	.7808265	.4735477	1.68	.7759650	.5895910
1.19	.7956513	.4914299	1.69	.8020001	.6174806
1.20	.8106556	.5094613	1.70	.8283154	.6456319
1.21	.8258410	.5276436	1.71	.8549136	.6740479
1.22	.8412089	.5459788	1.72	.8817974	.7027311
1.23	.8567610	.5644685	1.73	.9089692	.7316847
1.24	.8724988	.5831146	1.74	.9364319	.7609115
1.25	.8884239	.6019191	1.75	.9641884	.7904143
1.26	.9045378	.6208837	1.76	2.9922411	.8201962
1.27	.9208421	.6400105	1.77	3.0205932	.8502601
1.28	.9373385	.6593012	1.78	.0492473	.8806091
1.29	.9540287	.6787578	1.79	.0782063	.9112461
1.30	.9709143	.6983824	1.80	.1074732	.9421742
1.31	1.9879969	.7181768	1.81	.1370508	2.9733966
1.32	2.0052783	.7381431	1.82	.1669421	3.0049163
1.33	.0227603	.7582830	1.83	.1971501	.0367365
1.34	.0404446	.7785989	1.84	.2276799	.0688603
1.35	.0583329	.7990926	1.85	.2585283	.1012911
1.36	.0764271	.8197662	1.86	.2897047	.1340321
1.37	.0947288	.8406219	1.87	.3212100	.1670863
1.38	.1132401	.8616615	1.88	.3530475	.2004573
1.39	.1319627	.8828874	1.89	.3852202	.2341484
1.40	.1508985	.9043015	1.90	.4177315	.2681629
1.41	.1700494	.9259060	1.91	.4505846	.3025041
1.42	.1894172	.9477032	1.92	.4837827	.3371758
1.43	.2090041	.9696951	1.93	.5173293	.3721810
1.44	.2288118	1.9918840	1.94	.5512275	.4075235
1.45	.2488424	2.0142721	1.95	.5854808	.4432067
1.46	.2690979	.0368616	1.96	.6200927	.4792343
1.47	.2895803	.0596549	1.97	.6550667	.5156097
1.48	.3102917	.0826540	1.98	.6904061	.5523368
1.49	.3312341	.1058614	1.99	.7261146	.5894191
1.50	2.3524096	2.1292794	2.00	3.7621957	3.6268604

3

x	Cosh x	Sinh x	x	Cosh x	Sinh x
2.01	3.7986528	3.6646642	2.51	6.1930993	6.1118311
2.02	.8354899	.7028345	2.52	.2545281	.1740685
2.03	.8727101	.7413746	2.53	.3165827	.2369237
2.04	.9103184	.7802896	2.54	.3792687	.3004023
2.05	.9483548	.8196198	2.55	.4425928	.3645111
2.06	.9867111	.8592571	2.56	.5065611	.4292563
2.07	4.0255038	.8993179	2.57	.5711800	.4946444
2.08	.0647395	.9398093	2.58	.6364560	.5606820
2.09	.1043012	.9806140	2.59	.7023958	.6273758
2.10	.1443131	4.0218567	2.60	.7690059	.6947323
2.11	.1847398	.0635018	2.61	.8362940	.7627595
2.12	.2255846	.1055530	2.62	.9042644	.8314615
2.13	.2668523	.1480149	2.63	.9729254	.9008469
2.14	.3085462	.1908914	2.64	7.0422838	.9709225
2.15	.3506713	.2341871	2.65	.1123463	7.0416950
2.16	.3932312	.2779062	2.66	.1831184	.1131701
2.17	.4362311	.3220534	2.67	.2546108	.1853586
2.18	.4796741	.3666325	2.68	.3268282	.2582650
2.19	.5235649	.4116482	2.69	.3997785	.3318975
2.20	.5679083	.4571052	2.70	.4734686	.4062631
2.21	.6127086	.5030079	2.71	.5479060	.4813692
2.22	.6579702	.5493610	2.72	.6230984	.5572237
2.23	.7036972	.5961688	2.73	.6990531	.6338338
2.24	.7498951	.6434364	2.74	.7757775	.7112072
2.25	.7965677	.6911685	2.75	.8532799	.7893520
2.26	.8437197	.7393692	2.76	.9315674	.8682756
2.27	.8913565	.7880444	2.77	8.0106482	.9479862
2.28	.9394824	.8371982	2.78	.0905297	8.0284911
2.29	.9881022	.8868358	2.79	.1712205	.1097993
2.30	5.0372206	.9369618	2.80	.2527285	.1919185
2.31	.0868429	.9875817	2.81	.3350617	.2748566
2.32	.1369741	5.0387004	2.82	.4182283	.3586224
2.33	.1876186	.0903228	2.83	.5022368	.4432239
2.34	.2387822	.1424545	2.84	.5870956	.5286699
2.35	.2905196	.1951504	2.85	.6728130	.6149687
2.36	.3426859	.2482656	2.86	.7593979	.7021291
2.37	.3954365	.3019558	2.87	.8468585	.7901595
2.38	.4487266	.3561760	2.88	.9352041	.8790694
2.39	.5025618	.4109321	2.89	9.0244430	.9688668
2.40	.5569472	.4662293	2.90	.1145844	9.0595611
2.41	.6118883	.5220729	2.91	.2056373	.1511616
2.42	.6673910	.5784683	2.92	.2976105	.2436769
2.43	.7234594	.6354226	2.93	.3905138	.3371168
2.44	.7801009	.6929401	2.94	.4843559	.4314902
2.45	.8373201	.7510265	2.95	.5791467	.5268070
2.46	.8951232	.8096882	2.96	.6748952	.6230763
2.47	.9535159	.8689310	2.97	.7716115	.7203081
2.48	6.0125038	.9287605	2.98	.8693047	.8185119
2.49	.0720930	.9891831	2.99	.9679850	.9176976
2.50	6.1322895	6.0502045	3.00	10.0676620	10.0178750

4

x	Cosh x	Sinh x	x	Cosh x	Sinh x
3.01	10.1683456	10.1190539	3.51	16.7390823	16.7091854
3.02	.2700464	.2212451	3.52	.9070139	.8774144
3.03	.3727741	.3244585	3.53	17.0766361	17.0473312
3.04	.4765391	.4287042	3.54	.2479662	.2189529
3.05	.5813518	.5339929	3.55	.4210213	.3922966
3.06	.6872224	.6403347	3.56	.5958178	.5673790
3.07	.7941620	.7477408	3.57	.7723744	.7442186
3.08	.9021809	.8562217	3.58	.9507082	.9228325
3.09	11.0112900	.9657881	3.59	18.1308371	18.1032388
3.10	.1215004	11.0764511	3.60	.3127790	.2854552
3.11	.2328226	.1882217	3.61	.4965523	.4695004
3.12	.3452684	.3011112	3.62	.6821753	.6553927
3.13	.4588488	.4151309	3.63	.8696665	.8431503
3.14	.5735748	.5302919	3.64	19.0590447	19.0327924
3.15	.6894584	.6466062	3.65	.2503288	.2243376
3.16	.8065107	.7640850	3.66	.4435377	.4178052
3.17	.9247440	.8827403	3.67	.6386909	.6132145
3.18	12.0441695	12.0025838	3.68	.8358083	.8105854
3.19	.1647998	.1236279	3.69	20.0349094	20.0099373
3.20	.2866462	.2458839	3.70	.2360140	.2112905
3.21	.4097213	.3693646	3.71	.4391421	.4146645
3.22	.5340375	.4940825	3.72	.6443142	.6200802
3.23	.6596073	.6200497	3.73	.8515505	.8275577
3.24	.7864428	.7472790	3.74	21.0608720	21.0371178
3.25	.9145572	.8757829	3.75	.2722997	.2487819
3.26	13.0439629	13.0055744	3.76	.4858548	.4625710
3.27	.1746730	.1366665	3.77	.7015584	.6785064
3.28	.3067006	.2690723	3.78	.9194324	.8966096
3.29	.4400587	.4028048	3.79	22.1394981	22.1169025
3.30	.5747611	.5378780	3.80	.3617777	.3394069
3.31	.7108208	.6743046	3.81	.5862933	.5641452
3.32	.8482516	.8120988	3.82	.8130681	.7911403
3.33	.9870673	.9512741	3.83	23.0421239	23.0204143
3.34	14.1272820	14.0918450	3.84	.2734843	.2519907
3.35	.2689091	.2338247	3.85	.5071715	.4858917
3.36	.4119630	.3772277	3.86	.7432095	.7221415
3.37	.5564583	.5220686	3.87	.9816222	.9607638
3.38	.7024094	.6683619	3.88	24.2224327	24.2017819
3.39	.8498306	.8161219	3.89	.4656658	.4452205
3.40	.9987366	.9653634	3.90	.7113454	.6911034
3.41	15.1491429	15.1161016	3.91	.9594963	.9394557
3.42	.3010637	.2683513	3.92	25.2101431	25.1903020
3.43	.4545147	.4221278	3.93	.4633109	.4436673
3.44	.6095114	.5774468	3.94	.7190247	.6995765
3.45	.7660688	.7343232	3.95	.9773109	.9580561
3.46	.9242033	.8927735	3.96	26.2381943	26.2191311
3.47	16.0839298	16.0528128	3.97	.5017019	.4828285
3.48	.2452646	.2144571	3.98	.7678597	.7491740
3.49	.4082241	.3777233	3.99	27.0366943	27.0181946
3.50	16.5728248	16.5426275	4.00	.3082331	.2899175

APPENDIX II

LOGARITHMS OF NUMBERS

1

LOGARITHMS OF NUMBERS, FROM 0 TO 1000

No.	0	1	2	3	4	5	6	7	8	9
0	0	00000	30103	47712	60206	69897	77815	84510	90309	95424
10	00000	00432	00860	01283	01703	02118	02530	02938	03342	03742
11	04139	04532	04921	05307	05690	06069	06445	06818	07188	07554
12	07918	08278	08636	08990	09342	09691	10037	10380	10721	11059
13	11394	11727	12057	12385	12710	13033	13353	13672	13987	14301
14	14613	14921	15228	15533	15836	16136	16435	16731	17026	17318
15	17609	17897	18184	18469	18752	19033	19312	19590	19865	20139
16	20412	20682	20951	21218	21484	21748	22010	22271	22530	22788
17	23045	23299	23552	23804	24054	24303	24551	24797	25042	25285
18	25527	25767	26007	26245	26481	26717	26951	27184	27415	27646
19	27875	28103	28330	28555	28780	29003	29225	29446	29666	29885
20	30103	30319	30535	30749	30963	31175	31386	31597	31806	32014
21	32222	32428	32633	32838	33041	33243	33445	33646	33845	34044
22	34242	34439	34635	34830	35024	35218	35410	35602	35793	35983
23	36173	36361	36548	36735	36921	37106	37291	37474	37657	37839
24	38021	38201	38381	38560	38739	38916	39093	39269	39445	39619
25	39794	39967	40140	40312	40483	40654	40824	40993	41162	41330
26	41497	41664	41830	41995	42160	42324	42488	42651	42813	42975
27	43136	43296	43456	43616	43775	43933	44090	44248	44404	44560
28	44716	44870	45024	45178	45331	45484	45636	45788	45939	46089
29	46240	46389	46538	46686	46834	46982	47129	47275	47421	47567
30	47712	47856	48000	48144	48287	48430	48572	48713	48855	48995
31	49136	49276	49415	49554	49693	49831	49968	50105	50242	50379
32	50515	50650	50785	50920	51054	51188	51321	51454	51587	51719
33	51851	51982	52113	52244	52374	52504	52633	52763	52891	53020
34	53148	53275	53402	53529	53655	53781	53907	54033	54157	54282
35	54407	54530	54654	54777	54900	55022	55145	55266	55388	55509
36	55630	55750	55870	55990	56110	56229	56348	56466	56584	56702
37	56820	56937	57054	57170	57287	57403	57518	57634	57749	57863
38	57978	58092	58206	58319	58433	58546	58658	58771	58883	58995
39	59106	59217	59328	59439	59549	59659	59769	59879	59988	60097
40	60206	60314	60422	60530	60638	60745	60852	60959	61066	61172
41	61278	61384	61489	61595	61700	61804	61909	62013	62118	62221
42	62325	62428	62531	62634	62736	62838	62941	63042	63144	63245
43	63347	63447	63548	63648	63749	63848	63948	64048	64147	64246
44	64345	64443	64542	64640	64738	64836	64933	65030	65127	65224
45	65321	65417	65513	65609	65705	65801	65896	65991	66086	66181
46	66276	66370	66464	66558	66651	66745	66838	66931	67024	67117
47	67210	67302	67394	67486	67577	67669	67760	67851	67942	68033
48	68124	68214	68304	68394	68484	68574	68663	68752	68842	68930
49	69020	69108	69196	69284	69372	69460	69548	69635	69722	69810
50	69897	69983	70070	70156	70243	70329	70415	70500	70586	70671
51	70757	70842	70927	71011	71096	71180	71265	71349	71433	71516
52	71600	71683	71767	71850	71933	72015	72098	72181	72263	72345
53	72428	72509	72591	72672	72754	72835	72916	72997	73078	73158
54	73239	73319	73399	73480	73559	73639	73719	73798	73878	73957

APPENDIX III

TRIGONOMETRIC FUNCTIONS

2

LOGARITHMS OF NUMBERS, FROM 0 TO 1000

(Continued)

No.	0	1	2	3	4	5	6	7	8	9
55	74036	74115	74193	74272	74351	74429	74507	74585	74663	74741
56	74818	74896	74973	75050	75127	75204	75281	75358	75434	75511
57	75587	75663	75739	75815	75891	75966	76042	76117	76192	76267
58	76342	76417	76492	76566	76641	76715	76789	76863	76937	77011
59	77085	77158	77232	77305	77378	77451	77524	77597	77670	77742
60	77815	77887	77959	78031	78103	78175	78247	78318	78390	78461
61	78533	78604	78675	78746	78816	78887	78958	79028	79098	79169
62	79239	79309	79379	79448	79518	79588	79657	79726	79796	79865
63	79934	80002	80071	80140	80208	80277	80345	80413	80482	80550
64	80618	80685	80753	80821	80888	80956	81023	81090	81157	81224
65	81291	81358	81424	81491	81557	81624	81690	81756	81822	81888
66	81954	82020	82085	82151	82216	82282	82347	82412	82477	82542
67	82607	82672	82736	82801	82866	82930	82994	83058	83123	83187
68	83250	83314	83378	83442	83505	83569	83632	83695	83758	83821
69	83884	83947	84010	84073	84136	84198	84260	84323	84385	84447
70	84509	84571	84633	84695	84757	84818	84880	84941	85003	85064
71	85125	85187	85248	85309	85369	85430	85491	85551	85612	85672
72	85733	85793	85853	85913	85973	86033	86093	86153	86213	86272
73	86332	86391	86451	86510	86569	86628	86687	86746	86805	86864
74	86923	86981	87040	87098	87157	87215	87273	87332	87390	87448
75	87506	87564	87621	87679	87737	87794	87852	87909	87966	88024
76	88081	88138	88195	88252	88309	88366	88422	88479	88536	88592
77	88649	88705	88761	88818	88874	88930	88986	89042	89098	89153
78	89209	89265	89320	89376	89431	89487	89542	89597	89652	89707
79	89762	89817	89872	89927	89982	90036	90091	90145	90200	90254
80	90309	90363	90417	90471	90525	90579	90633	90687	90741	90794
81	90848	90902	90955	91009	91062	91115	91169	91222	91275	91328
82	91381	91434	91487	91540	91592	91645	91698	91750	91803	91855
83	91907	91960	92012	92064	92116	92168	92220	92272	92324	92376
84	92427	92479	92531	92582	92634	92685	92737	92788	92839	92890
85	92941	92993	93044	93095	93146	93196	93247	93298	93348	93399
86	93449	93500	93550	93601	93651	93701	93751	93802	93852	93902
87	93951	94001	94051	94101	94151	94200	94250	94300	94349	94398
88	94448	94497	94546	94596	94645	94694	94743	94792	94841	94890
89	94939	94987	95036	95085	95133	95182	95230	95279	95327	95376
90	95424	95472	95520	95568	95616	95664	95712	95760	95808	95856
91	95904	95951	95999	96047	96094	96142	96189	96236	96284	96331
92	96378	96426	96473	96520	96567	96614	96661	96708	96754	96801
93	96848	96895	96941	96988	97034	97081	97127	97174	97220	97266
94	97312	97359	97405	97451	97497	97543	97589	97635	97680	97726
95	97772	97818	97863	97909	97954	98000	98045	98091	98136	98181
96	98227	98272	98317	98362	98407	98452	98497	98542	98587	98632
97	98677	98721	98766	98811	98855	98900	98945	98989	99033	99078
98	99122	99166	99211	99255	99299	99343	99387	99431	99475	99519
99	99563	99607	99651	99694	99738	99782	99825	99869	99913	99956

APPENDIX III

TRIGONOMETRIC FUNCTIONS

1

NATURAL SINES, COSINES, TANGENTS, ETC.

°	′	Sine	Cosecant	Tangent	Cotangent	Secant	Cosine	′	°
0	0	.000000	Infinite	.000000	Infinite	1.00000	1.000000	0	**90**
	10	.002909	343.77516	.002909	343.77371	1.00000	.999996	50	
	20	.005818	171.88831	.005818	171.88540	1.00002	.999983	40	
	30	.008727	114.59301	.008727	114.58865	1.00004	.999962	30	
	40	.011635	85.945609	.011636	85.939791	1.00007	.999932	20	
	50	.014544	68.757360	.014545	68.750087	1.00011	.999894	10	
1	0	.017452	57.298688	.017455	57.289962	1.00015	.999848	0	**89**
	10	.020361	49.114062	.020365	49.103881	1.00021	.999793	50	
	20	.023269	42.975713	.023275	42.964077	1.00027	.999729	40	
	30	.026177	38.201550	.026186	38.188459	1.00034	.999657	30	
	40	.029085	34.382316	.029097	34.367771	1.00042	.999577	20	
	50	.031992	31.257577	.032009	31.241577	1.00051	.999488	10	
2	0	.034899	28.653708	.034921	28.636253	1.00061	.999391	0	**88**
	10	.037806	26.450510	.037834	26.431600	1.00072	.999285	50	
	20	.040713	24.562123	.040747	24.541758	1.00083	.999171	40	
	30	.043619	22.925586	.043661	22.903766	1.00095	.999048	30	
	40	.046525	21.493676	.046576	21.470401	1.00108	.998917	20	
	50	.049431	20.230284	.049491	20.205553	1.00122	.998778	10	
3	0	.052336	19.107323	.052408	19.081137	1.00137	.998630	0	**87**
	10	.055241	18.102619	.055325	18.074977	1.00153	.998473	50	
	20	.058145	17.198434	.058243	17.169337	1.00169	.998308	40	
	30	.061049	16.380408	.061163	16.349855	1.00187	.998135	30	
	40	.063952	15.636793	.064083	15.604784	1.00205	.997357	20	
	50	.066854	14.957882	.067004	14.924417	1.00224	.997763	10	
4	0	.069756	14.335587	.069927	14.300666	1.00244	.997564	0	**86**
	10	.072658	13.763115	.072851	13.726738	1.00265	.997357	50	
	20	.075559	13.234717	.075776	13.196888	1.00287	.997141	40	
	30	.078459	12.745495	.078702	12.706205	1.00309	.996917	30	
	40	.081359	12.291252	.081629	12.250505	1.00333	.996685	20	
	50	.084258	11.868370	.084558	11.826167	1.00357	.996444	10	
5	0	.087156	11.473713	.087489	11 430052	1.00382	.996195	0	**85**
	10	.090053	11.104549	.090421	11.059431	1.00408	.995937	50	
	20	.092950	10.758488	.093354	10.711913	1.00435	.995671	40	
	30	.095846	10.433431	.096289	10.385397	1.00463	.995396	30	
	40	.098741	10.127522	.099226	10.078031	1.00491	.995113	20	
	50	.101635	9.8391227	.102164	9.7881732	1.00521	.994822	10	
6	0	.104528	9.5667722	.105104	9.5143645	1.00551	.994522	0	**84**
	10	.107421	9.3091699	.108046	9.2553035	1.00582	.994214	50	
	20	.110313	9.0651512	.110990	9.0098261	1.00614	.993897	40	**83**
°	′	Cosine	Secant	Cotangent	Tangent	Cosecant	Sine	′	°

For functions from 83° 40′ to 90° read from bottom of table upward.

2

NATURAL SINES, COSINES, TANGENTS, ETC.
(*Continued*)

°	′	Sine	Cosecant	Tangent	Cotangent	Secant	Cosine	′	°
6	30	.113203	8.8336715	.113936	8.7768874	1.00647	.993572	30	
	40	.116093	8.6137901	.116883	8.5555468	1.00681	.993238	20	
	50	.118982	8.4045586	.119833	8.3449558	1.00715	.992896	10	
7	0	.121869	8.2055090	.122785	8.1443464	1.00751	.992546	0	83
	10	.124756	8.0156450	.125738	7.9530224	1.00787	.992187	50	
	20	.127642	7.8344335	.128694	7.7703506	1.00825	.991820	40	
	30	.130526	7.6612976	.131653	7.5957541	1.00863	.991445	30	
	40	.133410	7.4957100	.134613	7.4287064	1.00902	.991061	20	
	50	.136292	7.3371909	.137576	7.2687255	1.00942	.990669	10	
8	0	.139173	7.1852965	.140541	7.1153697	1.00983	.990268	0	82
	10	.142053	7.0396220	.143508	6.9682335	1.01024	.989859	50	
	20	.144932	6.8997942	.146478	6.8269437	1.01067	.989442	40	
	30	.147809	6.7654691	.149451	6.6911562	1.01111	.989016	30	
	40	.150686	6.6363293	.152426	6.5605538	1.01155	.988582	20	
	50	.153561	6.5120812	.155404	6.4348428	1.01200	.988139	10	
9	0	.156434	6.3924532	.158384	6.3137515	1.01247	.987688	0	81
	10	.159307	6.2771933	.161368	6.1970279	1.01294	.987229	50	
	20	.162178	6.1660674	.164354	6.0844381	1.01342	.986762	40	
	30	.165048	6.0588980	.167343	5.9757644	1.01391	.986286	30	
	40	.167916	5.9553625	.170334	5.8708042	1.01440	.985801	20	
	50	.170783	5.8553921	.173329	5.7693688	1.01491	.985309	10	
10	0	.173648	5.7587705	.176327	5.6712818	1.01543	.984808	0	80
	10	.176512	5.6653331	.179328	5.5763786	1.01595	.984298	50	
	20	.179375	5.5749258	.182332	5.4845052	1.01649	.983781	40	
	30	.182236	5.4874043	.185339	5.3955172	1.01703	.983255	30	
	40	.185095	5.4026333	.188359	5.3092793	1.01758	.982721	20	
	50	.187953	5.3204860	.191363	5.2256647	1.01815	.982178	10	
11	0	.190809	5.2408431	.194380	5.1445540	1.01872	.981627	0	79
	10	.193664	5.1635924	.197401	5.0658352	1.01930	.981068	50	
	20	.196517	5.0886284	.200425	4.9894027	1.01989	.980500	40	
	30	.199368	5.0158317	.203452	4.9151570	1.02049	.979925	30	
	40	.202218	4.9451687	.206483	4.8430045	1.02110	.979341	20	
	50	.205065	4.8764907	.209518	4.7728568	1.02171	.978748	10	
12	0	.207912	4.8097343	.212557	4.7046301	1.02234	.978148	0	78
	10	.210756	4.7448206	.215599	4.6382457	1.02298	.977539	50	
	20	.213599	4.6816748	.218645	4.5736287	1.02362	.976921	40	
	30	.216440	4.6202263	.221695	4.5107085	1.02428	.976296	30	
	40	.219279	4.5604080	.224748	4.4494181	1.02494	.975662	20	
	50	.222116	4.5021565	.227806	4.3896940	1.02562	.975020	10	77
°	′	Cosine	Secant	Cotangent	Tangent	Cosecant	Sine	′	°

For functions from 77° 10′ to 83° 30′ read from bottom of table upward.

3

NATURAL SINES, COSINES, TANGENTS, ETC.

(Continued)

°	′	Sine	Cosecant	Tangent	Cotangent	Secant	Cosine	′	°
13	0	.224951	4.4454115	.230868	4.3314759	1.02630	.974370	0	**77**
	10	.227784	4.3901158	.233934	4.2747066	1.02700	.973712	50	
	20	.230616	4.3362150	.237004	4.2193318	1.02770	.973045	40	
	30	.233445	4.2836576	.240079	4.1652998	1.02842	.972370	30	
	40	.236273	4.2323943	.243158	4.1125614	1.02914	.971687	20	
	50	.239098	4.1823785	.246241	4.0610700	1.02987	.970995	10	
14	0	.241922	4.1335655	.249328	4.0107809	1.03061	.970296	0	**76**
	10	.244743	4.0859130	.252420	3.9616518	1.03137	.969588	50	
	20	.247563	4.0393804	.255517	3.9136420	1.03213	.968872	40	
	30	.250380	3.9939292	.258618	3.8667131	1.03290	.968148	30	
	40	.253195	3.9495224	.261723	3.8208281	1.03363	.967415	20	
	50	.256008	3.9061250	.264834	3.7759519	1.03447	.966675	10	
15	0	.258819	3.8637033	.267949	3.7320508	1.03528	.965926	0	**75**
	10	.261628	3.8222251	.271069	3.6890927	1.03609	.965169	50	
	20	.264434	3.7816596	.274195	3.6470467	1.03691	.964404	40	
	30	.267238	3.7419775	.277325	3.6058835	1.03774	.963630	30	
	40	.270040	3.7031506	.280460	3.5655749	1.03858	.962849	20	
	50	.272840	3.6651518	.283600	3.5260938	1.03944	.962059	10	
16	0	.275637	3.6279553	.286745	3.4874144	1.04030	.961262	0	**74**
	10	.278432	3.5915363	.289896	3.4495120	1.04117	.960456	50	
	20	.281225	3.5558710	.293052	3.4123626	1.04206	.959642	40	
	30	.284015	3.5209365	.296214	3.3759434	1.04295	.958820	30	
	40	.286803	3.4867110	.299380	3.3402326	1.04385	.957990	20	
	50	.289589	3.4531735	.302553	3.3052091	1.04477	.957151	10	
17	0	.292372	3.4203036	.305731	3.2708526	1.04569	.956305	0	**73**
	10	.295152	3.3880820	.308914	3.2371438	1.04663	.955450	50	
	20	.297930	3.3564900	.312104	3.2040638	1.04757	.954588	40	
	30	.300706	3.3255095	.315299	3.1715948	1.04853	.953717	30	
	40	.303479	3.2951234	.318500	3.1397194	1.04950	.952838	20	
	50	.306249	3.2653149	.321707	3.1084210	1.05047	.951951	10	
18	0	.309017	3.2360680	.324920	3.0776835	1.05146	.951057	0	**72**
	10	.311782	3.2073673	.328139	3.0474915	1.05246	.950154	50	
	20	.314545	3.1791978	.331364	3.0178301	1.05347	.949243	40	
	30	.317305	3.1515453	.334595	2.9886850	1.05449	.948324	30	
	40	.320062	3.1243959	.337833	2.9600422	1.05552	.947397	20	
	50	.322816	3.0977363	.341077	2.9318885	1.05657	.946462	10	
19	0	.325568	3.0715535	.344328	2.9042109	1.05762	.945519	0	**71**
	10	.328317	3.0458352	.347585	2.8769970	1.05869	.944568	50	
	20	.331063	3.0205693	.350848	2.8502349	1.05976	.943609	40	**70**
°	′	Cosine	Secant	Cotangent	Tangent	Cosecant	Sine	′	°

For functions from 70° 40′ to 77° 0′ read from bottom of table upward.

4

NATURAL SINES, COSINES, TANGENTS, ETC.

(Continued)

°	′	Sine	Cosecant	Tangent	Cotangent	Secant	Cosine	′	°
19	30	.333807	2.9957443	.354119	2.8239129	1.06085	.942641	30	
	40	.336547	2.9713490	.357396	2.7980198	1.06195	.941666	20	
	50	.339285	2.9473724	.360680	2.7725448	1.06306	.940684	10	
20	0	.342020	2.9238044	.363970	2.7474774	1.06418	.939693	0	**70**
	10	.344752	2.9006346	.367268	2.7228076	1.06531	.938694	50	
	20	.347481	2.8778532	.370573	2.6985254	1.06645	.937687	40	
	30	.350207	2.8554510	.373885	2.6746215	1.06761	.936672	30	
	40	.352931	2.8334185	.377204	2.6510867	1.06878	.935650	20	
	50	.355651	2.8117471	.380530	2.6279121	1.06995	.934619	10	
21	0	.358368	2.7904281	.383864	2.6050891	1.07115	.933580	0	**69**
	10	.361082	2.7694532	.387205	2.5826094	1.07235	.932534	50	
	20	.363793	2.7488144	.390554	2.5604649	1.07356	.931480	40	
	30	.366501	2.7285038	.393911	2.5386479	1.07479	.930418	30	
	40	.369206	2.7085139	.397275	2.5171507	1.07602	.929348	20	
	50	.371908	2.6888374	.400647	2.4959661	1.07727	.928270	10	
22	0	.374607	2.6694672	.404026	2.4750869	1.07853	.927184	0	**68**
	10	.377302	2.6503962	.407414	2.4545061	1.07981	.926090	50	
	20	.379994	2.6316180	.410810	2.4342172	1.08109	.924989	40	
	30	.382683	2.6131259	.414214	2.4142136	1.08239	.923880	30	
	40	.385369	2.5949137	.417626	2.3944889	1.08370	.922762	20	
	50	.388052	2.5769753	.421046	2.3750372	1.08503	.921638	10	
23	0	.390731	2.5593047	.424475	2.3558524	1.08636	.920505	0	**67**
	10	.393407	2.5418961	.427912	2.3369287	1.08771	.919364	50	
	20	.396080	2.5247440	.431358	2.3182606	1.08907	.918216	40	
	30	.398749	2.5078428	.434812	2.2998425	1.09044	.917060	30	
	40	.401415	2.4911874	.438276	2.2816693	1.09183	.915896	20	
	50	.404078	2.4747726	.441748	2.2637357	1.09323	.914725	10	
24	0	.406737	2.4585933	.445229	2.2460368	1.09464	.913545	0	**66**
	10	.409392	2.4426448	.448719	2.2285676	1.09606	.912358	50	
	20	.412045	2.4269222	.452218	2.2113234	1.09750	.911164	40	
	30	.414693	2.4114210	.455726	2.1942997	1.09895	.909961	30	
	40	.417338	2.3961367	.459244	2.1774920	1.10041	.908751	20	
	50	.419980	2.3810650	.462771	2.1608958	1.10189	.907533	10	
25	0	.422618	2.3662016	.466308	2.1445069	1.10338	.906308	0	**65**
	10	.425253	2.3515424	.469854	2.1283213	1.10488	.905075	50	
	20	.427884	2.3370833	.473410	2.1123348	1.10640	.903834	40	
	30	.430511	2.3228205	.476976	2.0965436	1.10793	.902585	30	
	40	.433135	2.3087501	.480551	2.0809438	1.10947	.901329	20	
	50	.435755	2.2948685	.484137	2.0655318	1.11103	.900065	10	**64**
°	′	Cosine	Secant	Cotangent	Tangent	Cosecant	Sine	′	°

For functions from 64°-10′ to 70°-30′ read from bottom of table upward.

5

NATURAL SINES, COSINES, TANGENTS, ETC.

(Continued)

°	′	Sine	Cosecant	Tangent	Cotangent	Secant	Cosine	′	°
26	0	.438371	2.2811720	.487733	2.0503038	1.11260	.898794	0	**64**
	10	.440984	2.2676571	.491339	2.0352565	1.11419	.897515	50	
	20	.443593	2.2543204	.494955	2.0203862	1.11579	.896229	40	
	30	.446198	2.2411585	.498582	2.0056897	1.11740	.894934	30	
	40	.448799	2.2281681	.502219	1.9911637	1.11903	.893633	20	
	50	.451397	2.2153460	.505867	1.9768050	1.12067	.892323	10	
27	0	.453990	2.2026893	.509525	1.9626105	1.12233	.891007	0	**63**
	10	.456580	2.1901947	.513195	1.9485772	1.12400	.889682	50	
	20	.459166	2.1778595	.516876	1.9347020	1.12568	.888350	40	
	30	.461749	2.1656806	.520567	1.9209821	1.12738	.887011	30	
	40	.464327	2.1536553	.524270	1.9074147	1.12910	.885664	20	
	50	.466901	2.1417808	.527984	1.8939971	1.13083	.884309	10	
28	0	.469472	2.1300545	.531709	1.8807265	1.13257	.882948	0	**62**
	10	.472038	2.1184737	.535547	1.8676003	1.13433	.881578	50	
	20	.474600	2.1070359	.539195	1.8546159	1.13610	.880201	40	
	30	.477159	2.0957385	.542956	1.8417409	1.13789	.878817	30	
	40	.479713	2.0845792	.546728	1.8290628	1.13970	.877425	20	
	50	.482263	2.0735556	.550515	1.8164892	1.14152	.876026	10	
29	0	.484810	2.0626653	.554309	1.8040478	1.14335	.874620	0	**61**
	10	.487352	2.0519061	.558118	1.7917362	1.14521	.873206	50	
	20	.489890	2.0412757	.561939	1.7795524	1.14707	.871784	40	
	30	.492424	2.0307720	.565773	1.7674940	1.14896	.870356	30	
	40	.494953	2.0203929	.569619	1.7555590	1.15085	.868920	20	
	50	.497479	2.0101362	.573478	1.7437453	1.15277	.867476	10	
30	0	.500000	2.0000000	.577350	1.7320508	1.15470	.866025	0	**60**
	10	.502517	1.9899822	.581235	1.7204736	1.15665	.864567	50	
	20	.505030	1.9800810	.585134	1.7090116	1.15861	.863102	40	
	30	.507538	1.9702944	.589045	1.6976631	1.16059	.861629	30	
	40	.510043	1.9606206	.592970	1.6864261	1.16259	.860149	20	
	50	.512543	1.9510577	.596908	1.6752988	1.16460	.858662	10	
31	0	.515038	1.9416040	.600861	1.6642795	1.16663	.857167	0	**59**
	10	.517529	1.9322578	.604827	1.6533663	1.16868	.855665	50	
	20	.520016	1.9230173	.608807	1.6425576	1.17075	.854156	40	
	30	.522499	1.9138809	.612801	1.6318517	1.17283	.852640	30	
	40	.524977	1.9048469	.616809	1.6212469	1.17493	.851117	20	
	50	.527450	1.8959138	.620832	1.6107417	1.17704	.849586	10	
32	0	.529919	1.8870799	.624869	1.6003345	1.17918	.848048	0	**58**
	10	.532384	1.8783438	.628921	1.5900238	1.18133	.846503	50	
	20	.534844	1.8697040	.632988	1.5798079	1.18350	.844951	40	**57**
°	′	Cosine	Secant	Cotangent	Tangent	Cosecant	Sine	′	°

For functions from 57°-40′ to 64°-0′ read from bottom of table upward.

6

NATURAL SINES, COSINES, TANGENTS, ETC.

(*Continued*)

°	′	Sine	Cosecant	Tangent	Cotangent	Secant	Cosine	′	°
32	30	.537300	1.8611590	.637079	1.5696856	1.18569	.843391	30	
	40	.539751	1.8527073	.641167	1.5596552	1.18790	.841825	20	
	50	.542197	1.8443476	.645280	1.5497155	1.19012	.840251	10	
33	0	.544639	1.8360785	.649408	1.5398650	1.19236	.838671	0	57
	10	.547076	1.8278985	.653531	1.5301025	1.19463	.837083	50	
	20	.549509	1.8198065	.657710	1.5204261	1.19691	.835488	40	
	30	.551937	1.8118010	.661886	1.5108352	1.19920	.833886	30	
	40	.554360	1.8038809	.666077	1.5013282	1.20152	.832277	20	
	50	.556779	1.7960449	.670285	1.4919039	1.20386	830661	10	
34	0	.559193	1.7882916	.674509	1.4825610	1.20622	.829038	0	56
	10	.561602	1.7806201	.678749	1.4732983	1.20859	.827407	50	
	20	.564007	1.7730290	.683007	1.4641147	1.21099	.825770	40	
	30	.566406	1.7655173	.687281	1.4550090	1.21341	.824126	30	
	40	.568801	1.7580837	.691573	1.4459801	1.21584	.822475	20	
	50	.571191	1.7507273	.695881	1.4370268	1.21830	.820817	10	
35	0	.573576	1.7434468	.700208	1.4281480	1.22077	.819152	0	55
	10	.575957	1.7362413	.704552	1.4193427	1.22327	.817480	50	
	20	.578332	1.7291096	.708913	1.4106098	1.22579	.815801	40	
	30	.580703	1.7220508	.713293	1.4019483	1.22833	.814116	30	
	40	.583069	1.7150639	.717691	1.3933571	1.23089	.812423	20	
	50	.585429	1.7081478	.722108	1.3848355	1.23347	.810723	10	
36	0	.587785	1.7013016	.726543	1.3763810	1.23607	.809017	0	54
	10	.590136	1.6945244	.730996	1.3679959	1.23869	.807304	50	
	20	.592482	1.6878151	.735469	1.3596764	1.24134	.805584	40	
	30	.594823	1.6811730	.739961	1.3514224	1.24400	.803857	30	
	40	.597159	1.6745970	.744472	1.3432331	1.24669	.802123	20	
	50	.599489	1.6680864	.749003	1.3351075	1.24940	.800383	10	
37	0	.601815	1.6616401	.753554	1.3270448	1.25214	.798636	0	53
	10	.604136	1.6552575	.758125	1.3190441	1.25489	.796882	50	
	20	.606451	1.6489376	.762716	1.3111046	1.25767	.795121	40	
	30	.608761	1.6426796	.767627	1.3032254	1.26047	.793353	30	
	40	.611067	1.6364828	.771959	1.2954057	1.26330	.791579	20	
	50	.613367	1.6303462	.776612	1.2876447	1.26615	.789798	10	
38	0	.615661	1.6242692	.781286	1.2799416	1.26902	.788011	0	52
	10	.617951	1.6182510	.785981	1.2722957	1.27191	.786217	50	
	20	.620235	1.6122908	.790698	1.2647062	1.27483	.784416	40	
	30	.622515	1.6063879	.795436	1.2571723	1.27778	.782608	30	
	40	.624789	1.6005416	.800196	1.2496933	1.28075	.780794	20	
	50	.627057	1.5947511	.804080	1.2422685	1.28374	.778973	10	51
°	′	Cosine	Secant	Cotangent	Tangent	Cosecant	Sine	′	°

For functions from 51°-10′ to 57°-30′ read from bottom of table upward.

7

NATURAL SINES, COSINES, TANGENTS, ETC.

(Continued)

°	′	Sine	Cosecant	Tangent	Cotangent	Secant	Cosine	′	°
39	0	.629320	1.5890157	.809784	1.2348972	1.28676	.777146	0	**51**
	10	.631578	1.5833318	.814612	1.2275786	1.28980	.775312	50	
	20	.633831	1.5777077	.819463	1.2203121	1.29287	.773472	40	
	30	.636078	1.5721337	.824336	1.2130970	1.29597	.771625	30	
	40	.638320	1.5666121	.829234	1.2059327	1.29909	.769771	20	
	50	.640557	1.5611424	.834155	1.1988184	1.30223	.767911	10	
40	0	.642788	1.5557238	.839100	1.1917536	1.30541	.766044	0	**50**
	10	.645013	1.5503558	.844069	1.1847376	1.30861	.764171	50	
	20	.647233	1.5450378	.849062	1.1777698	1.31183	.762292	40	
	30	.649448	1.5397690	.854081	1.1708496	1.31509	.760406	30	
	40	.651657	1.5345491	.859124	1.1639763	1.31837	.758514	20	
	50	.653861	1.5293773	.864193	1.1571495	1.32168	.756615	10	
41	0	.656059	1.5242531	.869287	1.1503684	1.32501	.754710	0	**49**
	10	.658252	1.5191759	.874407	1.1436326	1.32838	.752798	50	
	20	.660439	1.5141452	.879553	1.1369414	1.33177	.750880	40	
	30	.662620	1.5091605	.884725	1.1302944	1.33519	.748956	30	
	40	.664796	1.5042211	.889924	1.1236909	1.33864	.747025	20	
	50	.666966	1.4993267	.895151	1.1171305	1.34212	.745088	10	
42	0	.669131	1.4944765	.900404	1.1106125	1.34563	.743145	0	**48**
	10	.671289	1.4896703	.905685	1.1041365	1.34917	.741195	50	
	20	.673443	1.4849073	.910994	1.0977020	1.35274	.739239	40	
	30	.675590	1.4801872	.916331	1.0913085	1.35634	.737277	30	
	40	.677732	1.4755095	.921697	1.0849554	1.35997	.735309	20	
	50	.679868	1.4708736	.927091	1.0786423	1.36363	.733335	10	
43	0	.681998	1.4662792	.932515	1.0723687	1.36733	.731354	0	**47**
	10	.684123	1.4617257	.937968	1.0661341	1.37105	.729367	50	
	20	.686242	1.4572127	.943451	1.0599381	1.37481	.727374	40	
	30	.688355	1.4527397	.948965	1.0537801	1.37860	.725374	30	
	40	.690462	1.4483063	.954508	1.0476598	1.38242	.723369	20	
	50	.692563	1.4439120	.960083	1.0415767	1.38628	.721357	10	
44	0	.694658	1.4395565	.965689	1.0355303	1.39016	.719340	0	**46**
	10	.696748	1.4352393	.971326	1.0295203	1.39409	.717316	50	
	20	.698832	1.4309602	.976996	1.0235461	1.39804	.715286	40	
	30	.700909	1.4267182	.982697	1.0176074	1.40203	.713251	30	
	40	.702981	1.4225134	.988432	1.0117088	1.40606	.711209	20	
	50	.705047	1.4183454	.994199	1.0058348	1.41012	.709161	10	
45	0	.707107	1.4142136	1.000000	1.0000000	1.41421	.707107	0	**45**
°	′	Cosine	Secant	Cotangent	Tangent	Cosecant	Sine	′	°

For functions from 45°-0′ to 51°-0′ read from bottom of table upward.

APPENDIX IV

SQUARES, CUBES, SQUARE ROOTS, ETC.

1

SQUARES, CUBES, SQUARE ROOTS, CUBE ROOTS, AND RECIPROCALS

No.	Squares	Cubes	Square Roots	Cube Roots	Reciprocals
1	1	1	1.0000000	1.0000000	1.000000000
2	4	8	1.4142136	1.2599210	.500000000
3	9	27	1.7320508	1.4422496	.333333333
4	16	64	2.0000000	1.5874011	.250000000
5	25	125	2.2360680	1.7099759	.200000000
6	36	216	2.4494897	1.8171206	.166666667
7	49	343	2.6457513	1.9129312	.142857143
8	64	512	2.8284271	2.0000000	.125000000
9	81	729	3.0000000	2.0800837	.111111111
10	100	1000	3.1622777	2.1544347	.100000000
11	121	1331	3.3166248	2.2239801	.090909091
12	144	1728	3.4641016	2.2894286	.083333333
13	169	2197	3.6055513	2.3513347	.076923077
14	196	2744	3.7416574	2.4101422	.071428571
15	225	3375	3.8729833	2.4662121	.066666667
16	256	4096	4.0000000	2.5198421	.062500000
17	289	4913	4.1231056	2.5712816	.058823529
18	324	5832	4.2426407	2.6207414	.055555556
19	361	6859	4.3588989	2.6684016	.052631579
20	400	8000	4.4721360	2.7144177	.050000000
21	441	9261	4.5825757	2.7589243	.047619048
22	484	10648	4.6904158	2.8020393	.045454545
23	529	12167	4.7958315	2.8438670	.043478261
24	576	13824	4.8989795	2.8844991	.041666667
25	625	15625	5.0000000	2.9240177	.040000000
26	676	17576	5.0990195	2.9624960	.038461538
27	729	19683	5.1961524	3.0000000	.037037037
28	784	21952	5.2915026	3.0365889	.035714286
29	841	24389	5.3851648	3.0723168	.034482759
30	900	27000	5.4772256	3.1072325	.033333333
31	961	29791	5.5677644	3.1413806	.032258065
32	1024	32768	5.6568542	3.1748021	.031250000
33	1089	35937	5.7445626	3.2075343	.030303030
34	1156	39304	5.8309519	3.2396118	.029411765
35	1225	42875	5.9160798	3.2710663	.028571429
36	1296	46656	6.0000000	3.3019272	.027777778
37	1369	50653	6.0827625	3.3322218	.027027027
38	1444	54872	6.1644140	3.3619754	.026315789
39	1521	59319	6.2449980	3.3912114	.025641026
40	1600	64000	6.3245553	3.4199519	.025000000
41	1681	68921	6.4031242	3.4482172	.024390244
42	1764	74088	6.4807407	3.4760266	.023809524
43	1849	79507	6.5574385	3.5033981	.023255814
44	1936	85184	6.6332496	3.5303483	.022727273
45	2025	91125	6.7082039	3.5568933	.022222222
46	2116	97336	6.7823300	3.5830479	.021739130
47	2209	103823	6.8556546	3.6088261	.021276596
48	2304	110592	6.9282032	3.6342411	.020833333
49	2401	117649	7.0000000	3.6593057	.020408163

2

SQUARES, CUBES, SQUARE ROOTS, CUBE ROOTS, AND RECIPROCALS

No.	Squares	Cubes	Square Roots	Cube Roots	Reciprocals
50	2500	125000	7.0710678	3.6840314	.020000000
51	2601	132651	7.1414284	3.7084298	.019607843
52	2704	140608	7.2111026	3.7325111	.019230769
53	2809	148877	7.2801099	3.7562858	.018867925
54	2916	157464	7.3484692	3.7797631	.018518519
55	3025	166375	7.4161985	3.8029525	.018181818
56	3136	175616	7.4833148	3.8258624	.017857143
57	3249	185193	7.5498344	3.8485011	.017543860
58	3364	195112	7.6157731	3.8708766	.017241379
59	3481	205379	7.6811457	3.8929965	.016949153
60	3600	216000	7.7459667	3.9148676	.016666667
61	3721	226981	7.8102497	3.9364972	.016393443
62	3844	238328	7.8740079	3.9578915	.016129032
63	3969	250047	7.9372539	3.9790571	.015873016
64	4096	262144	8.0000000	4.0000000	.015625000
65	4225	274625	8.0622577	4.0207256	.015384615
66	4356	287496	8.1240384	4.0412401	.015151515
67	4489	300763	8.1853528	4.0615480	.014925373
68	4624	314432	8.2462113	4.0816551	.014705882
69	4761	328509	8.3066239	4.1015661	.014492754
70	4900	343000	8.3666003	4.1212853	.014285714
71	5041	357911	8.4261498	4.1408178	.014084507
72	5184	373248	8.4852814	4.1601676	.013888889
73	5329	389017	8.5440037	4.1793390	.013698630
74	5476	405224	8.6023253	4.1983364	.013513514
75	5625	421875	8.6602540	4.2171633	.013333333
76	5776	438976	8.7177979	4.2358236	.013157895
77	5929	456533	8.7749644	4.2543210	.012987013
78	6084	474552	8.8317609	4.2726586	.012820513
79	6241	493039	8.8881944	4.2908404	.012658228
80	6400	512000	8.9442719	4.3088695	.012500000
81	6561	531441	9.0000000	4.3267487	.012345679
82	6724	551368	9.0553851	4.3444815	.012195122
83	6889	571787	9.1104336	4.3620707	.012048193
84	7056	592704	9.1651514	4.3795191	.011904762
85	7225	614125	9.2195445	4.3968296	.011764706
86	7396	636056	9.2736185	4.4140049	.011627907
87	7569	658503	9.3273791	4.4310476	.011494253
88	7744	681472	9.3808315	4.4479602	.011363636
89	7921	704969	9.4339811	4.4647451	.011235955
90	8100	729000	9.4868330	4.4814047	.011111111
91	8281	753571	9.5393920	4.4979414	.010989011
92	8464	778688	9.5916630	4.5143574	.010869565
93	8649	804357	9.6436508	4.5306549	.010752688
94	8836	830584	9.6953597	4.5468359	.010638298
95	9025	857375	9.7467943	4.5629026	.010526316
96	9216	884736	9.7979590	4.5788570	.010416667
97	9409	912673	9.8488578	4.5947009	.010309278
98	9604	941192	9.8994949	4.6104363	.010204082
99	9801	970299	9.9498744	4.6260650	.010101010

3

SQUARES, CUBES, SQUARE ROOTS, CUBE ROOTS, AND RECIPROCALS

No.	Squares	Cubes	Square Roots	Cube Roots	Reciprocals
100	10000	1000000	10.0000000	4.6415888	.010000000
101	10201	1030301	10.0498756	4.6570095	.009900990
102	10404	1061208	10.0995049	4.6723287	.009803922
103	10609	1092727	10.1488916	4.6875482	.009708738
104	10816	1124864	10.1980390	4.7026694	.009615385
105	11025	1157625	10.2469508	4.7176940	.009523810
106	11236	1191016	10.2956301	4.7326235	.009433962
107	11449	1225043	10.3440804	4.7474594	.009345794
108	11664	1259712	10.3923048	4.7622032	.009259259
109	11881	1295029	10.4403065	4.7768562	.009174312
110	12100	1331000	10.4880885	4.7914199	.009090909
111	12321	1367631	10.5356538	4.8058955	.009009009
112	12544	1404928	10.5830052	4.8202845	.008928571
113	12769	1442897	10.6301458	4.8345881	.008849558
114	12996	1481544	10.6770783	4.8488076	.008771930
115	13225	1520875	10.7238053	4.8629442	.008695652
116	13456	1560896	10.7703296	4.8769990	.008620690
117	13689	1601613	10.8166538	4.8909732	.008547009
118	13924	1643032	10.8627805	4.9048681	.008474576
119	14161	1685159	10.9087121	4.9186847	.008403361
120	14400	1728000	10.9544512	4.9324242	.008333333
121	14641	1771561	11.0000000	4.9460874	.008264463
122	14884	1815848	11.0453610	4.9596757	.008196721
123	15129	1860867	11.0905365	4.9731898	.008130081
124	15376	1906624	11.1355287	4.9866310	.008064516
125	15625	1953125	11.1803399	5.0000000	.008000000
126	15876	2000376	11.2249722	5.0132979	.007936508
127	16129	2048383	11.2694277	5.0265257	.007874016
128	16384	2097152	11.3137085	5.0396842	.007812500
129	16641	2146689	11.3578167	5.0527743	.007751938
130	16900	2197000	11.4017543	5.0657970	.007692308
131	17161	2248091	11.4455231	5.0787531	.007633588
132	17424	2299968	11.4891253	5.0916434	.007575758
133	17689	2352637	11.5325626	5.1044687	.007518797
134	17956	2406104	11.5758369	5.1172299	.007462687
135	18225	2460375	11.6189500	5.1299278	.007407407
136	18496	2515456	11.6619038	5.1425632	.007352941
137	18769	2571353	11.7046999	5.1551367	.007299270
138	19044	2628072	11.7473401	5.1676493	.007246377
139	19321	2685619	11.7898261	5.1801015	.007194245
140	19600	2744000	11.8321596	5.1924941	.007142857
141	19881	2803221	11.8743421	5.2048279	.007092199
142	20164	2863288	11.9163753	5.2171034	.007042254
143	20449	2924207	11.9582607	5.2293215	.006993007
144	20736	2985984	12.0000000	5.2414828	.006944444
145	21025	3048625	12.0415946	5.2535879	.006896552
146	21316	3112136	12.0830460	5.2656374	.006849315
147	21609	3176523	12.1243557	5.2776321	.006802721
148	21904	3241792	12.1655251	5.2895725	.006756757
149	22201	3307949	12.2065556	5.3014592	.006711409

4

SQUARES, CUBES, SQUARE ROOTS, CUBE ROOTS, AND RECIPROCALS

No.	Squares	Cubes	Square Roots	Cube Roots	Reciprocals
150	22500	3375000	12.2474487	5.3132928	.006666667
151	22801	3442951	12.2882057	5.3250740	.006622517
152	23104	3511808	12.3288280	5.3368033	.006578947
153	23409	3581577	12.3693169	5.3484812	.006535948
154	23716	3652264	12.4096736	5.3601084	.006493506
155	24025	3723875	12.4498996	5.3716854	.006451613
156	24336	3796416	12.4899960	5.3832126	.006410256
157	24649	3869893	12.5299641	5.3946907	.006369427
158	24964	3944312	12.5698051	5.4061202	.006329114
159	25281	4019679	12.6095202	5.4175015	.006289308
160	25600	4096000	12.6491106	5.4288352	.006250000
161	25921	4173281	12.6885775	5.4401218	.006211180
162	26244	4251528	12.7279221	5.4513618	.006172840
163	26569	4330747	12.7671453	5.4625556	.006134969
164	26896	4410944	12.8062485	5.4737037	.006097561
165	27225	4492125	12.8452326	5.4848066	.006060606
166	27556	4574296	12.8840987	5.4958647	.006024096
167	27889	4657463	12.9228480	5.5068784	.005988024
168	28224	4741632	12.9614814	5.5178484	.005952381
169	28561	4826809	13.0000000	5.5287748	.005917160
170	28900	4913000	13.0384048	5.5396583	.005882353
171	29241	5000211	13.0766968	5.5504991	.005847953
172	29584	5088448	13.1148770	5.5612978	.005813953
173	29929	5177717	13.1529464	5.5720546	.005780347
174	30276	5268024	13.1909060	5.5827702	.005747126
175	30625	5359375	13.2287566	5.5934447	.005714286
176	30976	5451776	13.2664992	5.6040787	.005681818
177	31329	5545233	13.3041347	5.6146724	.005649718
178	31684	5639752	13.3416641	5.6252263	.005617978
179	32041	5735339	13.3790882	5.6357408	.005586592
180	32400	5832000	13.4164079	5.6462162	.005555556
181	32761	5929741	13.4536240	5.6566528	.005524862
182	33124	6028568	13.4907376	5.6670511	.005494505
183	33489	6128487	13.5277493	5.6774114	.005464481
184	33856	6229504	13.5646600	5.6877340	.005434783
185	34225	6331625	13.6014705	5.6980192	.005405405
186	34596	6434856	13.6381817	5.7082675	.005376344
187	34969	6539203	13.6747943	5.7184791	.005347594
188	35344	6644672	13.7113092	5.7286543	.005319149
189	35721	6751269	13.7477271	5.7387936	.005291005
190	36100	6859000	13.7840488	5.7488971	.005263158
191	36481	6967871	13.8202750	5.7589652	.005235602
192	36864	7077888	13.8564065	5.7689982	.005208333
193	37249	7189057	13.8924440	5.7789966	.005181347
194	37636	7301384	13.9283883	5.7889604	.005154639
195	38025	7414875	13.9642400	5.7988900	.005128205
196	38416	7529536	14.0000000	5.8087857	.005102041
197	38809	7645373	14.0356688	5.8186479	.005076142
198	39204	7762392	14.0712473	5.8284767	.005050505
199	39601	7880599	14.1067360	5.8382725	.005025126

5

SQUARES, CUBES, SQUARE ROOTS, CUBE ROOTS, AND RECIPROCALS

No.	Squares	Cubes	Square Roots	Cube Roots	Reciprocals
200	40000	8000000	14.1421356	5.8480355	.005000000
201	40401	8120601	14.1774469	5.8577660	.004975124
202	40804	8242408	14.2126704	5.8674643	.004950495
203	41209	8365427	14.2478068	5.8771307	.004926108
204	41616	8489664	14.2828569	5.8867653	.004901961
205	42025	8615125	14.3178211	5.8963685	.004878049
206	42436	8741816	14.3527001	5.9059406	.004854369
207	42849	8869743	14.3874946	5.9154817	.004830918
208	43264	8998912	14.4222051	5.9249921	.004807692
209	43681	9129329	14.4568323	5.9344721	.004784689
210	44100	9261000	14.4913767	5.9439220	.004761905
211	44521	9393931	14.5258390	5.9533418	.004739336
212	44944	9528128	14.5602198	5.9627320	.004716981
213	45369	9663597	14.5945195	5.9720926	.004694836
214	45796	9800344	14.6287388	5.9814240	.004672897
215	46225	9938375	14.6628783	5.9907264	.004651163
216	46656	10077696	14.6969385	6.0000000	.004629630
217	47089	10218313	14.7309199	6.0092450	.004608295
218	47524	10360232	14.7648231	6.0184617	.004587156
219	47961	10503459	14.7986486	6.0276502	.004566210
220	48400	10648000	14.8323970	6.0368107	.004545455
221	48841	10793861	14.8660687	6.0459435	.004524887
222	49284	10941048	14.8996644	6.0550489	.004504505
223	49729	11089567	14.9331845	6.0641270	.004484305
224	50176	11239424	14.9666295	6.0731779	.004464286
225	50625	11390625	15.0000000	6.0822020	.004444444
226	51076	11543176	15.0332964	6.0911994	.004424779
227	51529	11697083	15.0665192	6.1001702	.004405286
228	51984	11852352	15.0996689	6.1091147	.004385965
229	52441	12008989	15.1327460	6.1180332	.004366812
230	52900	12167000	15.1657509	6.1269257	.004347826
231	53361	12326391	15.1986842	6.1357924	.004329004
232	53824	12487168	15.2315462	6.1446337	.004310345
233	54289	12649337	15.2643375	6.1534495	.004291845
234	54756	12812904	15.2970585	6.1622401	.004273504
235	55225	12977875	15.3297097	6.1710058	.004255319
236	55696	13144256	15.3622915	6.1797466	.004237288
237	56169	13312053	15.3948043	6.1884628	.004219409
238	56644	13481272	15.4272486	6.1971544	.004201681
239	57121	13651919	15.4596248	6.2058218	.004184100
240	57600	13824000	15.4919334	6.2144650	.004166667
241	58081	13997521	15.5241747	6.2230843	.004149378
242	58564	14172488	15.5563492	6.2316797	.004132231
243	59049	14348907	15.5884573	6.2402515	.004115226
244	59536	14526784	15.6204994	6.2487998	.004098361
245	60025	14706125	15.6524758	6.2573248	.004081633
246	60516	14886936	15.6843871	6.2658266	.004065041
247	61009	15069223	15.7162336	6.2743054	.004048583
248	61504	15252992	15.7480157	6.2827613	.004032258
249	62001	15438249	15.7797338	6.2911946	.004016064

6

SQUARES, CUBES, SQUARE ROOTS, CUBE ROOTS, AND RECIPROCALS

No.	Squares	Cubes	Square Roots	Cube Roots	Reciprocals
250	62500	15625000	15.8113883	6.2996053	.004000000
251	63001	15813251	15.8429795	6.3079935	.003984064
252	63504	16003008	15.8745079	6.3163596	.003968254
253	64009	16194277	15.9059737	6.3247035	.003952569
254	64516	16387064	15.9373775	6.3330256	.003937008
255	65025	16581375	15.9687194	6.3413257	.003921569
256	65536	16777216	16.0000000	6.3496042	.003906250
257	66049	16974593	16.0312195	6.3578611	.003891051
258	66564	17173512	16.0623784	6.3660968	.003875969
259	67081	17373979	16.0934769	6.3743111	.003861004
260	67600	17576000	16.1245155	6.3825043	.003846154
261	68121	17779581	16.1554944	6.3906765	.003831418
262	68644	17984728	16.1864141	6.3988279	.003816794
263	69169	18191447	16.2172747	6.4069585	.003802281
264	69696	18399744	16.2480768	6.4150687	.003787879
265	70225	18609625	16.2788206	6.4231583	.003773585
266	70756	18821096	16.3095064	6.4312276	.003759398
267	71289	19034163	16.3401346	6.4392767	.003745318
268	71824	19248832	16.3707055	6.4473057	.003731343
269	72361	19465109	16.4012195	6.4553148	.003717472
270	72900	19683000	16.4316767	6.4633041	.003703704
271	73441	19902511	16.4620776	6.4712736	.003690037
272	73984	20123648	16.4924225	6.4792236	.003676471
273	74529	20346417	16.5227116	6.4871541	.003663004
274	75076	20570824	16.5529454	6.4950653	.003649635
275	75625	20796875	16.5831240	6.5029572	.003636364
276	76176	21024576	16.6132477	6.5108300	.003623188
277	76729	21253933	16.6433170	6.5186839	.003610108
278	77284	21484952	16.6733320	6.5265189	.003597122
279	77841	21717639	16.7032931	6.5343351	.003584229
280	78400	21952000	16.7332005	6.5421326	.003571429
281	78961	22188041	16.7630546	6.5499116	.003558719
282	79524	22425768	16.7928556	6.5576722	.003546099
283	80089	22665187	16.8226038	6.5654144	.003533569
284	80656	22906304	16.8522995	6.5731385	.003521127
285	81225	23149125	16.8819430	6.5808443	.003508772
286	81796	23393656	16.9115345	6.5885323	.003496503
287	82369	23639903	16.9410743	6.5962023	.003484321
288	82944	23887872	16.9705627	6.6038545	.003472222
289	83521	24137569	17.0000000	6.6114890	.003460208
290	84100	24389000	17.0293864	6.6191060	.003448276
291	84681	24642171	17.0587221	6.6267054	.003436426
292	85264	24897088	17.0880075	6.6342874	.003424658
293	85849	25153757	17.1172428	6.6418522	.003412969
294	86436	25412184	17.1464282	6.6493998	.003401361
295	87025	25672375	17.1755640	6.6569302	003389831
296	87616	25934336	17.2046505	6.6644437	.003378378
297	88209	26198073	17.2336879	6.6719403	.003367003
298	88804	26463592	17.2626765	6.6794200	.003355705
299	89401	26730899	17.2916165	6.6868831	.003344482

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SQUARES, CUBES, SQUARE ROOTS, CUBE ROOTS, AND RECIPROCALS

No.	Squares	Cubes	Square Roots	Cube Roots	Reciprocals
300	90000	27000000	17.3205081	6.6943295	.003333333
301	90601	27270901	17.3493516	6.7017593	.003322259
302	91204	27543608	17.3781472	6.7091729	.003311258
303	91809	27818127	17.4068952	6.7165700	.003300330
304	92416	28094464	17.4355958	6.7239508	.003289474
305	93025	28372625	17.4642492	6.7313155	.003278681
306	93636	28652616	17.4928557	6.7386641	.003267974
307	94249	28934443	17.5214155	6.7459967	.003257329
308	94864	29218112	17.5499288	6.7533134	.003246753
309	95481	29503629	17.5783958	6.7606143	.003236246
310	96100	29791000	17.6068169	6.7678995	.003225806
311	96721	30080231	17.6351921	6.7751690	.003215434
312	97344	30371328	17.6635217	6.7824229	.003205128
313	97969	30664297	17.6918060	6.7896613	.003194888
314	98596	30959144	17.7200451	6.7968844	.003184713
315	99225	31255875	17.7482393	6.8040921	.003174603
316	99856	31554496	17.7763888	6.8112847	.003164557
317	100489	31855013	17.8044938	6.8184620	.003154574
318	101124	32157432	17.8325545	6.8256242	.003144654
319	101761	32461759	17.8605711	6.8327714	.003134796
320	102400	32768000	17.8885438	6.8399037	.003125000
321	103041	33076161	17.9164729	6.8470213	.003115265
322	103684	33386248	17.9443584	6.8541240	.003105590
323	104329	33698267	17.9722008	6.8612120	.003095975
324	104976	34012224	18.0000000	6.8682855	.003086420
325	105625	34328125	18.0277564	6.8753443	.003076923
326	106276	34645976	18.0554701	6.8823888	.003067485
327	106929	34965783	18.0831413	6.8894188	.003058104
328	107584	35287552	18.1107703	6.8964345	.003048780
329	108241	35611289	18.1383571	6.9034359	.003039514
330	108900	35937000	18.1659021	6.9104232	.003030303
331	109561	36264691	18.1934054	6.9173964	.003021148
332	110224	36594368	18.2208672	6.9243556	.003012048
333	110889	36926037	18.2482876	6.9313008	.003003003
334	111556	37259704	18.2756669	6.9382321	.002994012
335	112225	37595375	18.3030052	6.9451496	.002985075
336	112896	37933056	18.3303028	6.9520533	.002976190
337	113569	38272753	18.3575598	6.9589434	.002967359
338	114244	38614472	18.3847763	6.9658198	.002958580
339	114921	38958219	18.4119526	6.9726826	.002949853
340	115600	39304000	18.4390889	6.9795321	.002941176
341	116281	39651821	18.4661853	6.9863681	.002932551
342	116964	40001688	18.4932420	6.9931906	.002923977
343	117649	40353607	18.5202592	7.0000000	.002915452
344	118336	40707584	18.5472370	7.0067962	.002906977
345	119025	41063625	18.5741756	7.0135791	.002898551
346	119716	41421736	18.6010752	7.0203490	.002890173
347	120409	41781923	18.6279360	7.0271058	.002881844
348	121104	42144192	18.6547581	7.0338497	.002873563
349	121801	42508549	18.6815417	7.0405806	.002865330

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SQUARES, CUBES, SQUARE ROOTS, CUBE ROOTS, AND RECIPROCALS

No.	Squares	Cubes	Square Roots	Cube Roots	Reciprocals
350	122500	42875000	18.7082869	7.0472987	.002857143
351	123201	43243551	18.7349940	7.0540041	.002849003
352	123904	43614208	18.7616630	7.0606967	.002840909
353	124609	43986977	18.7882942	7.0673767	.002832861
354	125316	44361864	18.8148877	7.0740440	.002824859
355	126025	44738875	18.8414437	7.0806988	.002816901
356	126736	45118016	18.8679623	7.0873411	.002808989
357	127449	45499293	18.8944436	7.0939709	.002801120
358	128164	45882712	18.9208879	7.1005885	.002793296
359	128881	46268279	18.9472953	7.1071937	.002785515
360	129600	46656000	18.9736660	7.1137866	.002777778
361	130321	47045881	19.0000000	7.1203674	.002770083
362	131044	47437928	19.0262976	7.1269360	.002762431
363	131769	47832147	19.0525589	7.1334925	.002754821
364	132496	48228544	19.0787840	7.1400370	.002747253
365	133225	48627125	19.1049732	7.1465695	.002739726
366	133956	49027896	19.1311265	7.1530901	.002732240
367	134689	49430863	19.1572441	7.1595988	.002724796
368	135424	49836032	19.1833261	7.1660957	.002717391
369	136161	50243409	19.2093727	7.1725809	.002710027
370	136900	50653000	19.2353841	7.1790544	.002702703
371	137641	51064811	19.2613603	7.1855162	.002695418
372	138384	51478848	19.2873015	7.1919663	.002688172
373	139129	51895117	19.3132079	7.1984050	.002680965
374	139876	52313624	19.3390796	7.2048322	.002673797
375	140625	52734375	19.3649167	7.2112479	.002666667
376	141376	53157376	19.3907194	7.2176522	.002659574
377	142129	53582633	19.4164878	7.2240450	.002652520
378	142884	54010152	19.4422221	7.2304268	.002645503
379	143641	54439939	19.4679223	7.2367972	.002638522
380	144400	54872000	19.4935887	7.2431565	.002631579
381	145161	55306341	19.5192213	7.2495045	.002624672
382	145924	55742968	19.5448203	7.2558415	.002617801
383	146689	56181887	19.5703858	7.2621675	.002610966
384	147456	56623104	19.5959179	7.2684824	.002604167
385	148225	57066625	19.6214169	7.2747864	.002597403
386	148996	57512456	19.6468827	7.2810794	.002590674
387	149769	57960603	19.6723156	7.2873617	.002583979
388	150544	58411072	19.6977156	7.2936330	.002577320
389	151321	58863869	19.7230829	7.2998936	.002570694
390	152100	59319000	19.7484177	7.3061436	.002564103
391	152881	59776471	19.7737199	7.3123828	.002557545
392	153664	60236288	19.7989899	7.3186114	.002551020
393	154449	60698457	19.8242276	7.3248295	.002544529
394	155236	61162984	19.8494332	7.3310369	.002538071
395	156025	61629875	19.8746069	7.3372339	.002531646
396	156816	62099136	19.8997487	7.3434205	.002525253
397	157609	62570773	19.9248588	7.3495966	.002518892
398	158404	63044792	19.9499373	7.3557624	.002512563
399	159201	63521199	19.9749844	7.3619178	.002506266

SQUARES, CUBES, SQUARE ROOTS, CUBE ROOTS, AND RECIPROCALS

No.	Squares	Cubes	Square Roots	Cube Roots	Reciprocals
400	160000	64000000	20.0000000	7.3680630	.002500000
401	160801	64481201	20.0249844	7.3741979	.002493766
402	161604	64964808	20.0499377	7.3803227	.002487562
403	162409	65450827	20.0748599	7.3864373	.002481390
404	163216	65939264	20.0997512	7.3925418	.002475248
405	164025	66430125	20.1246118	7.3986362	.002469136
406	164836	66923416	20.1494417	7.4047206	.002463054
407	165649	67419143	20.1742410	7.4107950	.002457002
408	166464	67917312	20.1990099	7.4168595	.002450980
409	167281	68417929	20.2237484	7.4229142	.002444988
410	168100	68921000	20.2484567	7.4289589	.002439024
411	168921	69426531	20.2731349	7.4349938	.002433090
412	169744	69934528	20.2977831	7.4410189	.002427184
413	170569	70444997	20.3224014	7.4470342	.002421308
414	171396	70957944	20.3469899	7.4530399	.002415459
415	172225	71473375	20.3715488	7.4590359	.002409639
416	173056	71991296	20.3960781	7.4650223	.002403846
417	173889	72511713	20.4205779	7.4709991	.002398082
418	174724	73034632	20.4450483	7.4769664	.002392344
419	175561	73560059	20.4694895	7.4829242	.002386635
420	176400	74088000	20.4939015	7.4888724	.002380952
421	177241	74618461	20.5182845	7.4948113	.002375297
422	178084	75151448	20.5426386	7.5007406	.002369668
423	178929	75686967	20.5669638	7.5066607	.002364066
424	179776	76225024	20.5912603	7.5125715	.002358491
425	180625	76765625	20.6155281	7.5184730	.002352941
426	181476	77308776	20.6397674	7.5243652	.002347418
427	182329	77854483	20.6639783	7.5302482	.002341920
428	183184	78402752	20.6881609	7.5361221	.002336449
429	184041	78953589	20.7123152	7.5419867	.002331002
430	184900	79507000	20.7364414	7.5478423	.002325581
431	185761	80062991	20.7605395	7.5536888	.002320186
432	186624	80621568	20.7846097	7.5595263	.002314815
433	187489	81182737	20.8086520	7.5653548	.002309469
434	188356	81746504	20.8326667	7.5711743	.002304147
435	189225	82312875	20.8566536	7.5769849	.002298851
436	190096	82881856	20.8806130	7.5827865	.002293578
437	190969	83453453	20.9045450	7.5885793	.002288330
438	191844	84027672	20.9284495	7.5943633	.002283105
439	192721	84604519	20.9523268	7.6001385	.002277904
440	193600	85184000	20.9761770	7.6059049	.002272727
441	194481	85766121	21.0000000	7.6116626	.002267574
442	195364	86350888	21.0237960	7.6174116	.002262443
443	196249	86938307	21.0475652	7.6231519	.002257336
444	197136	87528384	21.0713075	7.6288837	.002252252
445	198025	88121125	21.0950231	7.6346067	.002247191
446	198916	88716536	21.1187121	7.6403213	.002242152
447	199809	89314623	21.1423745	7.6460272	.002237136
448	200704	89915392	21.1660105	7.6517247	.002232143
449	201601	90518849	21.1896201	7.6574138	.002227171

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SQUARES, CUBES, SQUARE ROOTS, CUBE ROOTS, AND RECIPROCALS

No.	Squares	Cubes	Square Roots	Cube Roots	Reciprocals
450	202500	91125000	21.2132034	7.6630943	.002222222
451	203401	91733851	21.2367606	7.6687665	.002217295
452	204304	92345408	21.2602916	7.6744303	.002212389
453	205209	92959677	21.2837967	7.6800857	.002207506
454	206116	93576664	21.3072758	7.6857328	.002202643
455	207025	94196375	21.3307290	7.6913717	.002197802
456	207936	94818816	21.3541565	7.6970023	.002192982
457	208849	95443993	21.3775583	7.7026246	.002188184
458	209764	96071912	21.4009346	7.7082388	.002183406
459	210681	96702579	21.4242853	7.7138448	.002178649
460	211600	97336000	21.4476106	7.7194426	.002173913
461	212521	97972181	21.4709106	7.7250325	.002169197
462	213444	98611128	21.4941853	7.7306141	.002164502
463	214369	99252847	21.5174348	7.7361877	.002159827
464	215296	99897344	21.5406592	7.7417532	.002155172
465	216225	100544625	21.5638587	7.7473109	.002150538
466	217156	101194696	21.5870331	7.7528606	.002145923
467	218089	101847563	21.6101828	7.7584023	.002141328
468	219024	102503232	21.6333077	7.7639361	.002136752
469	219961	103161709	21.6564078	7.7694620	.002132196
470	220900	103823000	21.6794834	7.7749801	.002127660
471	221841	104487111	21.7025344	7.7804904	.002123142
472	222784	105154048	21.7255610	7.7859928	.002118644
473	223729	105823817	21.7485632	7.7914875	.002114165
474	224676	106496424	21.7715411	7.7969745	.002109705
475	225625	107171875	21.7944947	7.8024538	.002105263
476	226576	107850176	21.8174242	7.8079254	.002100840
477	227529	108531333	21.8403297	7.8133892	.002096436
478	228484	109215352	21.8632111	7.8188456	.002092050
479	229441	109902239	21.8860686	7.8242942	.002087683
480	230400	110592000	21.9089023	7.8297353	.002083333
481	231361	111284641	21.9317122	7.8351688	.002079002
482	232324	111980168	21.9544984	7.8405949	.002074689
483	233289	112678587	21.9772610	7.8460134	.002070393
484	234256	113379904	22.0000000	7.8514244	.002066116
485	235225	114084125	22.0227155	7.8568281	.002061856
486	236196	114791256	22.0454077	7.8622242	.002057613
487	237169	115501303	22.0680765	7.8676130	.002053388
488	238144	116214272	22.0907220	7.8729944	.002049180
489	239121	116930169	22.1133444	7.8783684	.002044990
490	240100	117649000	22.1359436	7.8837352	.002040816
491	241081	118370771	22.1585198	7.8890946	.002036660
492	242064	119095488	22.1810730	7.8944468	.002032520
493	243049	119823157	22.2036033	7.8997917	.002028398
494	244036	120553784	22.2261108	7.9051294	.002024291
495	245025	121287375	22.2485955	7.9104599	.002020202
496	246016	122023936	22.2710575	7.9157832	.002016129
497	247009	122763473	22.2934968	7.9210994	.002012072
498	248004	123505992	22.3159136	7.9264085	.002008032
499	249001	124251499	22.3383079	7.9317104	.002004008

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SQUARES, CUBES, SQUARE ROOTS, CUBE ROOTS, AND RECIPROCALS

No.	Squares	Cubes	Square Roots	Cube Roots	Reciprocals
500	250000	125000000	22.3606798	7.9370053	.002000000
501	251001	125751501	22.3830293	7.9422931	.001996008
502	252004	126506008	22.4053565	7.9475739	.001992032
503	253009	127263527	22.4276615	7.9528477	.001988072
504	254016	128024064	22.4499443	7.9581144	.001984127
505	255025	128787625	22.4722051	7.9633743	.001980198
506	256036	129554216	22.4944438	7.9686271	.001976285
507	257049	130323843	22.5166605	7.9738731	.001972387
508	258064	131096512	22.5388553	7.9791122	.001968504
509	259081	131872229	22.5610283	7.9843444	.001964637
510	260100	132651000	22.5831796	7.9895697	.001960784
511	261121	133432831	22.6053091	7.9947883	.001956947
512	262144	134217728	22.6274170	8.0000000	.001953125
513	263169	135005697	22.6495033	8.0052049	.001949318
514	264196	135796744	22.6715681	8.0104032	.001945525
515	265225	136590875	22.6936114	8.0155946	.001941748
516	266256	137388096	22.7156334	8.0207794	.001937984
517	267289	138188413	22.7376340	8.0259574	.001934236
518	268324	138991832	22.7596134	8.0311287	.001930502
519	269361	139798359	22.7815715	8.0362935	.001926782
520	270400	140608000	22.8035085	8.0414515	.001923077
521	271441	141420761	22.8254244	8.0466030	.001919386
522	272484	142236648	22.8473193	8.0517479	.001915709
523	273529	143055667	22.8691933	8.0568862	.001912046
524	274576	143877824	22.8910463	8.0620180	.001908397
525	275625	144703125	22.9128785	8.0671432	.001904762
526	276676	145531576	22.9346899	8.0722620	.001901141
527	277729	146363183	22.9564806	8.0773743	.001897533
528	278784	147197952	22.9782506	8.0824800	.001893939
529	279841	148035889	23.0000000	8.0875794	.001890359
530	280900	148877000	23.0217289	8.0926723	.001886792
531	281961	149721291	23.0434372	8.0977589	.001883239
532	283024	150568768	23.0651252	8.1028390	.001879699
533	284089	151419437	23.0867928	8.1079128	.001876173
534	285156	152273304	23.1084400	8.1129803	.001872659
535	286225	153130375	23.1300670	8.1180414	.001869159
536	287296	153990656	23.1516738	8.1230962	.001865672
537	288369	154854153	23.1732605	8.1281447	.001862197
538	289444	155720872	23.1948270	8.1331870	.001858736
539	290521	156590819	23.2163735	8.1382230	.001855288
540	291600	157464000	23.2379001	8.1432529	.001851852
541	292681	158340421	23.2594067	8.1482765	.001848429
542	293764	159220088	23.2808935	8.1532939	.001845018
543	294849	160103007	23.3023604	8.1583051	.001841621
544	295936	160989184	23.3238076	8.1633102	.001838235
545	297025	161878625	23.3452351	8.1683092	.001834862
546	298116	162771336	23.3666429	8.1733020	.001831502
547	299209	163667323	23.3880311	8.1782888	.001828154
548	300304	164566592	23.4093998	8.1832695	.001824818
549	301401	165469149	23.4307490	8.1882441	.001821494

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SQUARES, CUBES, SQUARE ROOTS, CUBE ROOTS, AND RECIPROCALS

No.	Squares	Cubes	Square Roots	Cube Roots	Reciprocals
550	302500	166375000	23.4520788	8.1932127	.001818182
551	303601	167284151	23.4733892	8.1981753	.001814882
552	304704	168196608	23.4946802	8.2031319	.001811594
553	305809	169112377	23.5159520	8.2080825	.001808318
554	306916	170031464	23.5372046	8.2130271	.001805054
555	308025	170953875	23.5584380	8.2179657	.001801802
556	309136	171879616	23.5796522	8.2228985	.001798561
557	310249	172808693	23.6008474	8.2278254	.001795332
558	311364	173741112	23.6220236	8.2327463	.001792115
559	312481	174676879	23.6431808	8.2376614	.001788909
560	313600	175616000	23.6643191	8.2425706	.001785714
561	314721	176558481	23.6854386	8.2474740	.061782531
562	315844	177504328	23.7065392	8.2523715	.001779359
563	316969	178453547	23.7276210	8.2572633	.001776199
564	318096	179406144	23.7486842	8.2621492	.001773050
565	319225	180362125	23.7697286	8.2670294	.001769912
566	320356	181321496	23.7907545	8.2719039	.001766784
567	321489	182284263	23.8117618	8.2767726	.001763668
568	322624	183250432	23.8327506	8.2816355	.001760563
569	323761	184220009	23.8537209	8.2864928	.001757469
570	324900	185193000	23.8746728	8.2913444	.001754386
571	326041	186169411	23.8956063	8.2961903	.001751313
572	327184	187149248	23.9165215	8.3010304	.001748252
573	328329	188132517	23.9374184	8.3058651	.001745201
574	329476	189119224	23.9582971	8.3106941	.001742160
575	330625	190109375	23.9791576	8.3155175	.001739130
576	331776	191102976	24.0000000	8.3203353	.001736111
577	332929	192100033	24.0208243	8.3251475	.001733102
578	334084	193100552	24.0416306	8.3299542	.001730104
579	335241	194104539	24.0624188	8.3347553	.001727116
580	336400	195112000	24.0831891	8.3395509	.001724138
581	337561	196122941	24.1039416	8.3443410	.001721170
582	338724	197137368	24.1246762	8.3491256	.001718213
583	339889	198155287	24.1453929	8.3539047	.001715266
584	341056	199176704	24.1660919	8.3586784	.001712329
585	342225	200201625	24.1867732	8.3634466	.001709402
586	343396	201230056	24.2074369	8.3682095	.001706485
587	344569	202262003	24.2280829	8.3729668	.001703578
588	345744	203297472	24.2487113	8.3777188	.001700680
589	346921	204336469	24.2693222	8.3824653	.001697793
590	348100	205379000	24.2899156	8.3872065	.001694915
591	349281	206425071	24.3104916	8.3919423	.001692047
592	350464	207474688	24.3310501	8.3966729	.001689189
593	351649	208527857	24.3515913	8.4013981	.001686341
594	352836	209584584	24.3721152	8.4061180	.001683502
595	354025	210644875	24.3926218	8.4108326	.001680672
596	355216	211708736	24.4131112	8.4155419	.001677852
597	356409	212776173	24.4335834	8.4202460	.001675042
598	357604	213847192	24.4540385	8.4249448	.001672241
599	358801	214921799	24.4744765	8.4296383	.001669449

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SQUARES, CUBES, SQUARE ROOTS, CUBE ROOTS, AND RECIPROCALS

No.	Squares	Cubes	Square Roots	Cube Roots	Reciprocals
600	360000	216000000	24.4948974	8.4343267	.001666667
601	361201	217081801	24.5153013	8.4390098	.001663894
602	362404	218167208	24.5356883	8.4436877	.001661130
603	363609	219256227	24.5560583	8.4483605	.001658375
604	364816	220348864	24.5764115	8.4530281	.001655629
605	366025	221445125	24.5967478	8.4576906	.001652893
606	367236	222545016	24.6170673	8.4623479	.001650165
607	368449	223648543	24.6373700	8.4670001	.001647446
608	369664	224755712	24.6576560	8.4716471	.001644737
609	370881	225866529	24.6779254	8.4762892	.001642036
610	372100	226981000	24.6981781	8.4809261	.001639344
611	373321	228099131	24.7184142	8.4855579	.001636661
612	374544	229220928	24.7386338	8.4901848	.001633987
613	375769	230346397	24.7588368	8.4948065	.001631321
614	376996	231475544	24.7790234	8.4994233	.001628664
615	378225	232608375	24.7991935	8.5040350	.001626016
616	379456	233744896	24.8193473	8.5086417	.001623377
617	380689	234885113	24.8394847	8.5132435	.001620746
618	381924	236029032	24.8596058	8.5178403	.001618123
619	383161	237176659	24.8797106	8.5224321	.001615509
620	384400	238328000	24.8997992	8.5270189	.001612903
621	385641	239483061	24.9198716	8.5316009	.001610306
622	386884	240641848	24.9399278	8.5361780	.001607717
623	388129	241804367	24.9599679	8.5407501	.001605136
624	389376	242970624	24.9799920	8.5453173	.001602564
625	390625	244140625	25.0000000	8.5498797	.001600000
626	391876	245314376	25.0199920	8.5544372	.001597444
627	393129	246491883	25.0399681	8.5589899	.001594896
628	394384	247673152	25.0599282	8.5635377	.001592357
629	395641	248858189	25.0798724	8.5680807	.001589825
630	396900	250047000	25.0998008	8.5726189	.001587302
631	398161	251239591	25.1197134	8.5771523	.001584786
632	399424	252435968	25.1396102	8.5816809	.001582278
633	400689	253636137	25.1594913	8.5862047	.001579779
634	401956	254840104	25.1793566	8.5907238	.001577287
635	403225	256047875	25.1992063	8.5952380	.001574803
636	404496	257259456	25.2190404	8.5997476	.001572327
637	405769	258474853	25.2388589	8.6042525	.001569859
638	407044	259694072	25.2586619	8.6087526	.001567398
639	408321	260917119	25.2784493	8.6132480	.001564945
640	409600	262144000	25.2982213	8.6177388	.001562500
641	410881	263374721	25.3179778	8.6222248	.001560062
642	412164	264609288	25.3377189	8.6267063	.001557632
643	413449	265847707	25.3574447	8.6311830	.001555210
644	414736	267089984	25.3771551	8.6356551	.001552795
645	416025	268336125	25.3968502	8.6401226	.001550388
646	417316	269586136	25.4165301	8.6445855	.001547988
647	418609	270840023	25.4361947	8.6490437	.001545595
648	419904	272097792	25.4558441	8.6534974	.001543210
649	421201	273359449	25.4754784	8.6579465	.001540832

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SQUARES, CUBES, SQUARE ROOTS, CUBE ROOTS, AND RECIPROCALS

No.	Squares	Cubes	Square Roots	Cube Roots	Reciprocals
650	422500	274625000	25.4950976	8.6623911	.001538462
651	423801	275894451	25.5147016	8.6668310	.001536098
652	425104	277167808	25.5342907	8.6712665	.001533742
653	426409	278445077	25.5538647	8.6756974	.001531394
654	427716	279726264	25.5734237	8.6801237	.001529052
655	429025	281011375	25.5929678	8.6845456	.001526718
656	430336	282300416	25.6124969	8.6889630	.001524390
657	431649	283593393	25.6320112	8.6933759	.001522070
658	432964	284890312	25.6515107	8.6977843	.001519757
659	434281	286191179	25.6709953	8.7021882	.001517451
660	435600	287496000	25.6904652	8.7065877	.001515152
661	436921	288804781	25.7099203	8.7109827	.001512859
662	438244	290117528	25.7293607	8.7153734	.001510574
663	439569	291434247	25.7487864	8.7197596	.001508296
664	440896	292754944	25.7681975	8.7241414	.001506024
665	442225	294079625	25.7875939	8.7285187	.001503759
666	443556	295408296	25.8069758	8.7328918	.001501502
667	444889	296740963	25.8263431	8.7372604	.001499250
668	446224	298077632	25.8456960	8.7416246	.001497006
669	447561	299418309	25.8650343	8.7459846	.001494768
670	448900	300763000	25.8843582	8.7503401	.001492537
671	450241	302111711	25.9036677	8.7546913	.001490313
672	451584	303464448	25.9229628	8.7590383	.001488095
673	452929	304821217	25.9422435	8.7633809	.001485884
674	454276	306182024	25.9615100	8.7677192	.001483680
675	455625	307546875	25.9807621	8.7720532	.001481481
676	456976	308915776	26.0000000	8.7763830	.001479290
677	458329	310288733	26.0192237	8.7807084	.001477105
678	459684	311665752	26.0384331	8.7850296	.001474926
679	461041	313046839	26.0576284	8.7893466	.001472754
680	462400	314432000	26.0768096	8.7936593	.001470588
681	463761	315821241	26.0959767	8.7979679	.001468429
682	465124	317214568	26.1151297	8.8022721	.001466276
683	466489	318611987	26.1342687	8.8065722	.001464129
684	467856	320013504	26.1533937	8.8108681	.001461988
685	469225	321419125	26.1725047	8.8151598	.001459854
686	470596	322828856	26.1916017	8.8194474	.001457726
687	471969	324242703	26.2106848	8.8237307	.001455604
688	473344	325660672	26.2297541	8.8280099	.001453488
689	474721	327082769	26.2488095	8.8322850	.001451379
690	476100	328509000	26.2678511	8.8365559	.001449275
691	477481	329939371	26.2868789	8.8408227	.001447178
692	478864	331373888	26.3058929	8.8450854	.001445087
693	480249	332812557	26.3248932	8.8493440	.001443001
694	481636	334255384	26.3438797	8.8535985	.001440922
695	483025	335702375	26.3628527	8.8578489	.001438849
696	484416	337153536	26.3818119	8.8620952	.001436782
697	485809	338608873	26.4007576	8.8663375	.001434720
698	487204	340068392	26.4196896	8.8705757	.001432665
699	488601	341532099	26.4386081	8.8748099	.001430615

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SQUARES, CUBES, SQUARE ROOTS, CUBE ROOTS, AND RECIPROCALS

No.	Squares	Cubes	Square Roots	Cube Roots	Reciprocals
700	490000	343000000	26.4575131	8.8790400	.001428571
701	491401	344472101	26.4764046	8.8832661	.001426534
702	492804	345948408	26.4952826	8.8874882	.001424501
703	494209	347428927	26.5141472	8.8917063	.001422475
704	495616	348913664	26.5329983	8.8959204	.001420455
705	497025	350402625	26.5518361	8.9001304	.001418440
706	498436	351895816	26.5706605	8.9043366	.001416431
707	499849	353393243	26.5894716	8.9085387	.001414427
708	501264	354894912	26.6082694	8.9127369	.001412429
709	502681	356400829	26.6270539	8.9169311	.001410437
710	504100	357911000	26.6458252	8.9211214	.001408451
711	505521	359425431	26.6645833	8.9253078	.001406470
712	506944	360944128	26.6833281	8.9294902	.001404494
713	508369	362467097	26.7020598	8.9336687	.001402525
714	509796	363994344	26.7207784	8.9378433	.001400560
715	511225	365525875	26.7394839	8.9420140	.001398601
716	512656	367061696	26.7581763	8.9461809	.001396648
717	514089	368601813	26.7768557	8.9503438	.001394700
718	515524	370146232	26.7955220	8.9545029	.001392758
719	516961	371694959	26.8141754	8.9586581	.001390821
720	518400	373248000	26.8328157	8.9628095	.001388889
721	519841	374805361	26.8514432	8.9669570	.001386963
722	521284	376367048	26.8700577	8.9711007	.001385042
723	522729	377933067	26.8886593	8.9752406	.001383126
724	524176	379503424	26.9072481	8.9793766	.001381215
725	525625	381078125	26.9258240	8.9835089	.001379310
726	527076	382657176	26.9443872	8.9876373	.001377410
727	528529	384240583	26.9629375	8.9917620	.001375516
728	529984	385828352	26.9814751	8.9958829	.001373626
729	531441	387420489	27.0000000	9.0000000	.001371742
730	532900	389017000	27.0185122	9.0041134	.001369863
731	534361	390617891	27.0370117	9.0082229	.001367989
732	535824	392223168	27.0554985	9.0123288	.001366120
733	537289	393832837	27.0739727	9.0164309	.001364256
734	538756	395446904	27.0924344	9.0205293	.001362398
735	540225	397065375	27.1108834	9.0246239	.001360544
736	541696	398688256	27.1293199	9.0287149	.001358696
737	543169	400315553	27.1477439	9.0328021	.001356852
738	544644	401947272	27.1661554	9.0368857	.001355014
739	546121	403583419	27.1845544	9.0409655	.001353180
740	547600	405224000	27.2029410	9.0450417	.001351351
741	549081	406869021	27.2213152	9.0491142	.001349528
742	550564	408518488	27.2396769	9.0531831	.001347709
743	552049	410172407	27.2580263	9.0572482	.001345895
744	553536	411830784	27.2763634	9.0613098	.001344086
745	555025	413493625	27.2946881	9.0653677	.001342282
746	556516	415160936	27.3130006	9.0694220	.001340483
747	558009	416832723	27.3313007	9.0734726	.001338688
748	559504	418508992	27.3495887	9.0775197	.001336898
749	561001	420189749	27.3678644	9.0815631	.001335113

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SQUARES, CUBES, SQUARE ROOTS, CUBE ROOTS, AND RECIPROCALS

No.	Squares	Cubes	Square Roots	Cube Roots	Reciprocals
750	562500	421875000	27.3861279	9.0856030	.001333333
751	564001	423564751	27.4043792	9.0896392	.001331558
752	565504	425259008	27.4226184	9.0936719	.001329787
753	567009	426957777	27.4408455	9.0977010	.001328021
754	568516	428661064	27.4590604	9.1017265	.001326260
755	570025	430368875	27.4772633	9.1057485	.001324503
756	571536	432081216	27.4954542	9.1097669	.001322751
757	573049	433798093	27.5136330	9.1137818	.001321004
758	574564	435519512	27.5317998	9.1177931	.001319261
759	576081	437245479	27.5499546	9.1218010	.001317523
760	577600	438976000	27.5680975	9.1258053	.001315789
761	579121	440711081	27.5862284	9.1298061	.001314060
762	580644	442450728	27.6043475	9.1338034	.001312336
763	582169	444194947	27.6224546	9.1377971	.001310616
764	583696	445943744	27.6405499	9.1417874	.001308901
765	585225	447697125	27.6586334	9.1457742	.001307190
766	586756	449455096	27.6767050	9.1497576	.001305483
767	588289	451217663	27.6947648	9.1537375	.001303781
768	589824	452984832	27.7128129	9.1577139	.001302083
769	591361	454756609	27.7308492	9.1616869	.001300390
770	592900	456533000	27.7488739	9.1656565	.001298701
771	594441	458314011	27.7668868	9.1696225	.001297017
772	595984	460099648	27.7848880	9.1735852	.001295337
773	597529	461889917	27.8028775	9.1775445	.001293661
774	599076	463684824	27.8208555	9.1815003	.001291990
775	600625	465484375	27.8388218	9.1854527	.001290323
776	602176	467288576	27.8567766	9.1894018	.001288660
777	603729	469097433	27.8747197	9.1933474	.001287001
778	605284	470910952	27.8926514	9.1972897	.001285347
779	606841	472729139	27.9105715	9.2012286	.001283697
780	608400	474552000	27.9284801	9.2051641	.001282051
781	609961	476379541	27.9463772	9.2090962	.001280410
782	611524	478211768	27.9642629	9.2130250	.001278772
783	613089	480048687	27.9821372	9.2169505	.001277139
784	614656	481890304	28.0000000	9.2208726	.001275510
785	616225	483736625	28.0178515	9.2247914	.001273885
786	617796	485587656	28.0356915	9.2287068	.001272265
787	619369	487443403	28.0535203	9.2326189	.001270648
788	620944	489303872	28.0713377	9.2365277	.001269036
789	622521	491169069	28.0891438	9.2404333	.001267427
790	624100	493039000	28.1069386	9.2443355	.001265823
791	625681	494913671	28.1247222	9.2482344	.001264223
792	627264	496793088	28.1424946	9.2521300	.001262626
793	628849	498677257	28.1602557	9.2560224	.001261034
794	630436	500566184	28.1780056	9.2599114	.001259446
795	632025	502459875	28.1957444	9.2637973	.001257862
796	633616	504358336	28.2134720	6.2676798	.001256281
797	635209	506261573	28.2311884	9.2715592	.001254705
798	636804	508169592	28.2488938	9.2754352	.001253133
799	638401	510082399	28.2665881	9.2793081	.001251564

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SQUARES, CUBES, SQUARE ROOTS, CUBE ROOTS, AND RECIPROCALS

No.	Squares	Cubes	Square Roots	Cube Roots	Reciprocals
800	640000	512000000	28.2842712	9.2831777	.001250000
801	641601	513922401	28.3019434	9.2870440	.001248439
802	643204	515849608	28.3196045	9.2909072	.001246883
803	644809	517781627	28.3372546	9.2947671	.001245330
804	646416	519718464	28.3548938	9.2986239	.001243781
805	648025	521660125	28.3725219	9.3024775	.001242236
806	649636	523606616	28.3901391	9.3063278	.001240695
807	651249	525557943	28.4077454	9.3101750	.001239157
808	652864	527514112	28.4253408	9.3140190	.001237624
809	654481	529475129	28.4429253	9.3178599	.001236094
810	656100	531441000	28.4604989	9.3216975	.001234568
811	657721	533411731	28.4780617	9.3255320	.001233046
812	659344	535387328	28.4956137	9.3293634	.001231527
813	660969	537367797	28.5131549	9.3331916	.001230012
814	662596	539353144	28.5306852	9.3370167	.001228501
815	664225	541343375	28.5482048	9.3408386	.001226994
816	665856	543338496	28.5657137	9.3446575	.001225490
817	667489	545338513	28.5832119	9.3484731	.001223990
818	669124	547343432	28.6006993	9.3522857	.001222494
819	670761	549353259	28.6181760	9.3560952	.001221001
820	672400	551368000	28.6356421	9.3599016	.001219512
821	674041	553387661	28.6530976	9.3637049	.001218027
822	675684	555412248	28.6705424	9.3675051	.001216545
823	677329	557441767	28.6879766	9.3713022	.001215067
824	678976	559476224	28.7054002	9.3750963	.001213592
825	680625	561515625	28.7228132	9.3788873	.001212121
826	682276	563559976	28.7402157	9.3826752	.001210654
827	683929	565609283	28.7576077	9.3864600	.001209190
828	685584	567663552	28.7749891	9.3902419	.001207729
829	687241	569722789	28.7923601	9.3940206	.001206273
830	688900	571787000	28.8097206	9.3977964	.001204819
831	690561	573856191	28.8270706	9.4015691	.001203369
832	692224	575930368	28.8444102	9.4053387	.001201923
833	693889	578009537	28.8617394	9.4091054	.001200480
834	695556	580093704	28.8790582	9.4128690	.001199041
835	697225	582182875	28.8963666	9.4166297	.001197605
836	698896	584277056	28.9136646	9.4203873	.001196172
837	700569	586376253	28.9309523	9.4241420	.001194743
838	702244	588480472	28.9482297	9.4278936	.001193317
839	703921	590589719	28.9654967	9.4316423	.001191895
840	705600	592704000	28.9827535	9.4353880	.001190476
841	707281	594823321	29.0000000	9.4391307	.001189061
842	708964	596947688	29.0172363	9.4428704	.001187648
843	710649	599077107	29.0344623	9.4466072	.001186240
844	712336	601211584	29.0516781	9.4503410	.001184834
845	714025	603351125	29.0688837	9.4540719	.001183432
846	715716	605495736	29.0860791	9.4577999	.001182033
847	717409	607645423	29.1032644	9.4615249	.001180638
848	719104	609800192	29.1204396	9.4652470	.001179245
849	720801	611960049	29.1376046	9.4689661	.001177856

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SQUARES, CUBES, SQUARE ROOTS, CUBE ROOTS, AND RECIPROCALS

No.	Squares	Cubes	Square Roots	Cube Roots	Reciprocals
850	722500	614125000	29.1547595	9.4726824	.001176471
851	724201	616295051	29.1719043	9.4763957	.001175088
852	725904	618470208	29.1890390	9.4801061	.001173709
853	727609	620650477	29.2061637	9.4838136	.001172333
854	729316	622835864	29.2232784	9.4875182	.001170960
855	731025	625026375	29.2403830	9.4912200	.001169591
856	732736	627222016	29.2574777	9.4949188	.001168224
857	734449	629422793	29.2745623	9.4986147	.001166861
858	736164	631628712	29.2916370	9.5023078	.001165501
859	737881	633839779	29.3087018	9.5059980	.001164144
860	739600	636056000	29.3257566	9.5096854	.001162791
861	741321	638277381	29.3428015	9.5133699	.001161440
862	743044	640503928	29.3598365	9.5170515	.001160093
863	744769	642735647	29.3768616	9.5207303	.001158749
864	746496	644972544	29.3938769	9.5244063	.001157407
865	748225	647214625	29.4108823	9.5280794	.001156069
866	749956	649461896	29.4278779	9.5317497	.001154734
867	751689	651714363	29.4448637	9.5354172	.001153403
868	753424	653972032	29.4618397	9.5390818	.001152074
869	755161	656234909	29.4788059	9.5427437	.001150748
870	756900	658503000	29.4957624	9.5464027	.001149425
871	758641	660776311	29.5127091	9.5500589	.001148106
872	760384	663054848	29.5296461	9.5537123	.001146789
873	762129	665338617	29.5465734	9.5573630	.001145475
874	763876	667627624	29.5634910	9.5610108	.001144165
875	765625	669921875	29.5803989	9.5646559	.001142857
876	767376	672221376	29.5972972	9.5682982	.001141553
877	769129	674526133	29.6141858	9.5719377	.001140251
878	770884	676836152	29.6310648	9.5755745	.001138952
879	772641	679151439	29.6479342	9.5792085	.001137656
880	774400	681472000	29.6647939	9.5828397	.001136364
881	776161	683797841	29.6816442	9.5864682	.001135074
882	777924	686128968	29.6984848	9.5900939	.001133787
883	779689	688465387	29.7153159	9.5937169	.001132503
884	781456	690807104	29.7321375	9.5973373	.001131222
885	783225	693154125	29.7489496	9.6009548	.001129944
886	784996	695506456	29.7657521	9.6045696	.001128668
887	786769	697864103	29.7825452	9.6081817	.001127396
888	788544	700227072	29.7993289	9.6117911	.001126126
889	790321	702595369	29.8161030	9.6153977	.001124859
890	792100	704969000	29.8328678	9.6190017	.001123596
891	793881	707347971	29.8496231	9.6226030	.001122334
892	795664	709732288	29.8663690	9.6262016	.001121076
893	797449	712121957	29.8831056	9.6297975	.001119821
894	799236	714516984	29.8998328	9.6333907	.001118568
895	801025	716917375	29.9165506	9.6369812	.001117318
896	802816	719323136	29.9332591	9.6405690	.001116071
897	804609	721734273	29.9499583	9.6441542	.001114827
898	806404	724150792	29.9666481	9.6477367	.001113586
899	808201	726572699	29.9833287	9.6513166	.001112347

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SQUARES, CUBES, SQUARE ROOTS, CUBE ROOTS, AND RECIPROCALS

No.	Squares	Cubes	Square Roots	Cube Roots	Reciprocals
900	810000	729000000	30.0000000	9.6548938	.001111111
901	811801	731432701	30.0166620	9.6584684	.001109878
902	813604	733870808	30.0333148	9.6620403	.001108647
903	815409	736314327	30.0499584	9.6656096	.001107420
904	817216	738763264	30.0665928	9.6691762	.001106195
905	819025	741217625	30.0832179	9.6727403	.001104972
906	820836	743677416	30.0998339	9.6763017	.001103753
907	822649	746142643	30.1164407	9.6798604	.001102536
908	824464	748613312	30.1330383	9.6834166	.001101322
909	826281	751089429	30.1496269	9.6869701	.001100110
910	828100	753571000	30.1662063	9.6905211	.001098901
911	829921	756058031	30.1827765	9.6940694	.001097695
912	831744	758550528	30.1993377	9.6976151	.001096491
913	833569	761048497	30.2158899	9.7011583	.001095290
914	835396	763551944	30.2324329	9.7046989	.001094092
915	837225	766060875	30.2489669	9.7082369	.001092896
916	839056	768575296	30.2654919	9.7117723	.001091703
917	840889	771095213	30.2820079	9.7153051	.001090513
918	842724	773620632	30.2985148	9.7188354	.001089325
919	844561	776151559	30.3150128	9.7223631	.001088139
920	846400	778688000	30.3315018	9.7258883	.001086957
921	848241	781229961	30.3479818	9.7294109	.001085776
922	850084	783777448	30.3644529	9.7329309	.001084599
923	851929	786330467	30.3809151	9.7364484	.001083424
924	853776	788889024	30.3973683	9.7399634	.001082251
925	855625	791453125	30.4138127	9.7434758	.001081081
926	857476	794022776	30.4302481	9.7469857	.001079914
927	859329	796597983	30.4466747	9.7504930	.001078749
928	861184	799178752	30.4630924	9.7539979	.001077586
929	863041	801765089	30.4795013	9.7575002	.001076426
930	864900	804357000	30.4959014	9.7610001	.001075269
931	866761	806954491	30.5122926	9.7644974	.001074114
932	868624	809557568	30.5286750	9.7679922	.001072961
933	870489	812166237	30.5450487	9.7714845	.001071811
934	872356	814780504	30.5614136	9.7749743	.001070664
935	874225	817400375	30.5777697	9.7784616	.001069519
936	876096	820025856	30.5941171	9.7819466	.001068376
937	877969	822656953	30.6104557	9.7854288	.001067236
938	879844	825293672	30.6267857	9.7889087	.001066098
939	881721	827936019	30.6431069	9.7923861	.001064963
940	883600	830584000	30.6594194	9.7958611	.001063830
941	885481	833237621	30.6757233	9.7993336	.001062699
942	887364	835896888	30.6920185	9.8028036	.001061571
943	889249	838561807	30.7083051	9.8062711	.001060445
944	891136	841232384	30.7245830	9.8097362	.001059322
945	893025	843908625	30.7408523	9.8131989	.001058201
946	894916	846590536	30.7571130	9.8166591	.001057082
947	896809	849278123	30.7733651	9.8201169	.001055966
948	898704	851971392	30.7896086	9.8235723	.001054852
949	900601	854670349	30.8058436	9.8270252	.001053741

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SQUARES, CUBES, SQUARE ROOTS, CUBE ROOTS, AND RECIPROCALS

No.	Squares	Cubes	Square Roots	Cube Roots	Reciprocals
950	902500	857375000	30.8220700	9.8304757	.001052632
951	904401	860085351	30.8382879	9.8339288	.001051525
952	906304	862801408	30.8544972	9.8373695	.001050420
953	908209	865523177	30.8706981	9.8408127	.001049318
954	910116	868250664	30.8868904	9.8442536	.001048218
955	912025	870983875	30.9030743	9.8476920	.001047120
956	913936	873722816	30.9192497	9.8511280	.001046025
957	915849	876467493	30.9354166	9.8545617	.001044932
958	917764	879217912	30.9515751	9.8579929	.001043841
959	919681	881974079	30.9677251	9.8614218	.001042753
960	921600	884736000	30.9838668	9.8648483	.001041667
961	923521	887503681	31.0000000	9.8682724	.001040583
962	925444	890277128	31.0161248	9.8716941	.001039501
963	927369	893056347	31.0322413	9.8751135	.001038422
964	929296	895841344	31.0483494	9.8785305	.001037344
965	931225	898632125	31.0644491	9.8819451	.001036269
966	933156	901428696	31.0805405	9.8853574	.001035197
967	935089	904231063	31.0966236	9.8887673	.001034126
968	937024	907039232	31.1126984	9.8921749	.001033058
969	938961	909853209	31.1287648	9.8955801	.001031992
970	940900	912673000	31.1448230	9.8989830	.001030928
971	942841	915498611	31.1608729	9.9023835	.001029866
972	944784	918330048	31.1769145	9.9057817	.001028807
973	946729	921167317	31.1929479	9.9091776	.001027749
974	948676	924010424	31.2089731	9.9125712	.001026694
975	950625	926859375	31.2249900	9.9159624	.001025641
976	952576	929714176	31.2409987	9.9193513	.001024590
977	954529	932574833	31.2569992	9.9227379	.001023541
978	956484	935441352	31.2729915	9.9261222	.001022495
979	958441	938313739	31.2889757	9.9295042	.001021450
980	960400	941192000	31.3049517	9.9328839	.001020408
981	962361	944076141	31.3209195	9.9362613	.001019368
982	964324	946966168	31.3368792	9.9396363	.001018330
983	966289	949862087	31.3528308	9.9430092	.001017294
984	968256	952763904	31.3687743	9.9463797	.001016260
985	970225	955671625	31.3847097	9.9497479	.001015228
986	972196	958585256	31.4006369	9.9531138	.001014199
987	974169	961504803	31.4165561	9.9564775	.001013171
988	976144	964430272	31.4324673	9.9598389	.001012146
989	978121	967361669	31.4483704	9.9631981	.001011122
990	980100	970299000	31.4642654	9.9665549	.001010101
991	982081	973242271	31.4801525	9.9699095	.001009082
992	984064	976191488	31.4960315	9.9732619	.001008065
993	986049	979146657	31.5119025	9.9766120	.001007049
994	988036	982107784	31.5277655	9.9799599	.001006036
995	990025	985074875	31.5436206	9.9833055	.001005025
996	992016	988047936	31.5594677	9.9866488	.001004016
997	994009	991026973	31.5753068	9.9899900	.001003009
998	996004	994011992	31.5911380	9.9933289	.001002004
999	998001	997002999	31.6069613	9.9966656	.001001001

APPENDIX V

CONVERSION TABLES

1

TABLES FOR CONVERTING UNITED STATES WEIGHTS AND MEASURES

METRIC TO CUSTOMARY

WEIGHTS

No.	Milligrams to Grains	Grams to Troy Ounces	Grams to Avoirdupois Ounces	Kilograms to Avoirdupois Pounds	Tonnes to Net Tons of 2000 Pounds	Tonnes to Gross Tons of 2240 Pounds
1	.01543	.03215	.03527	2.20462	1.10231	.98421
2	.03086	.06430	.07055	4.40924	2.20462	1.96841
3	.04630	.09645	.10582	6.61387	3.30693	2.95262
4	.06173	.12860	.14110	8.81849	4.40924	3.93682
5	.07716	.16075	.17637	11.02311	5.51156	4.92103
6	.09259	.19290	.21164	13.22773	6.61387	5.90524
7	.10803	.22506	.24692	15.43236	7.71618	6.88944
8	.12346	.25721	.28219	17.63698	8.81849	7.87365
9	.13889	.28936	.31747	19.84160	9.92080	8.85785

1 Kilogram = 15432.35639 Grains

LINEAR MEASURE

No.	Millimeters to 64ths of an Inch	Centimeters to Inches	Meters to Feet	Meters to Yards	Kilometers to Statute Miles	Kilometers to Nautical Miles
1	2.51968	.39370	3.280833	1.093611	.62137	.53959
2	5.03936	.78740	6.561667	2.187222	1.24274	1.07919
3	7.55904	1.18110	9.842500	3.280833	1.86411	1.61878
4	10.07872	1.57480	13.123333	4.374444	2.48548	2.15837
5	12.59840	1.96850	16.404167	5.468056	3.10685	2.69796
6	15.11808	2.36220	19.685000	6.561667	3.72822	3.23756
7	17.63776	2.75590	22.965833	7.655278	4.34959	3.77715
8	20.15744	3.14960	26.246667	8.748889	4.97096	4.31674
9	22.67712	3.54330	29.527500	9.842500	5.59233	4.85633

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TABLES FOR CONVERTING UNITED STATES WEIGHTS AND MEASURES

CUSTOMARY TO METRIC

WEIGHTS

No.	Grains to Milligrams	Troy Ounces to Grams	Avoirdupois Ounces to Grams	Avoirdupois Pounds to Kilograms	Net Tons of 2000 Pounds to Tonnes	Gross Tons of 2240 Pounds to Tonnes
1	64.79892	31.10348	28.34953	.45359	.90718	1.01605
2	129.59784	62.20696	56.69905	.90718	1.81437	2.03209
3	194.39675	93.31044	85.04858	1.36078	2.72155	3.04814
4	259.19567	124.41392	113.39811	1.81437	3.62874	4.06419
5	323.99459	155.51740	141.74763	2.26796	4.53592	5.08024
6	388.79351	186.62088	170.09716	2.72155	5.44311	6.09628
7	453.59243	217.72437	198.44669	3.17515	6.35029	7.11233
8	518.39135	248.82785	226.79621	3.62874	7.25748	8.12838
9	583.19026	279.93133	255.14574	4.08233	8.16466	9.14442

1 Avoirdupois Pound = 453.5924277 Grams

LINEAR MEASURE

No.	64ths of an Inch to Millimeters	Inches to Centimeters	Feet to Meters	Yards to Meters	Statute Miles to Kilometers	Nautical Miles to Kilometers
1	.39688	2.54001	.304801	.914402	1.60935	1.85325
2	.79375	5.08001	.609601	1.828804	3.21869	3.70650
3	1.19063	7.62002	.914402	2.743205	4.82804	5.55975
4	1.58750	10.16002	1.219202	3.657607	6.43739	7.41300
5	1.98438	12.70003	1.524003	4.572009	8.04674	9.26625
6	2.38125	15.24003	1.828804	5.486411	9.65608	11.11950
7	2.77813	17.78004	2.133604	6.400813	11.26543	12.97275
8	3.17501	20.32004	2.438405	7.315215	12.87478	14.82600
9	3.57188	22.86005	2.743205	8.229616	14.48412	16.67925

1 Nautical Mile = 1853.25 Meters
1 Gunter's Chain = 20.1168 Meters
1 Fathom = 1.829 Meters

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